

BLACK HOLES AND THEIR ENTROPY

A Dissertation

by

JIANWEI MEI

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2010

Major Subject: Physics

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ABSTRACT

Black Holes and Their Entropy. (August 2010)

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This dissertation covers two different but related topics: the construction of new black hole solutions and the study of the microscopic origin of black hole entropy.

In the solution part, two different sets of new solutions are found. The first concerns a Plebanski-Demianski type solution in the five-dimensional pure Einstein gravity, and the second concerns a three-charge (two of which equal) two-rotation solution to the five-dimensional maximal supergravity. Obtaining new and interesting black hole solutions is an important and challenging task in studying general relativity and its extensions. During the past decade, the solutions become even more important because they might find applications in the study of the gauge/gravity duality, which is currently in the central stage of the quantum gravity research.

The Kerr/CFT correspondence is a recently propose example of the gauge/gravity duality. In the entropy part, we explicitly show that the Kerr/CFT correspondence can be applied to all known extremal stationary and axisymmetric black holes. We improve over previous works in showing that this can be done in a general fashion, rather than testing different solutions case by case. This effort makes it obvious that the common structure of the near horizon metric for all known extremal stationary and axisymmetric black holes is playing a key role in the success of the Kerr/CFT correspondence. The discussion is made possible by the identification of two general ansätze that cover all such known solutions.

To my family

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CHAPTER I

INTRODUCTION

It is commonly believed that all the currently known phenomena are governed by only four fundamental interactions: gravity, electromagnetism, the weak and the strong forces. The last three are described by gauge theories based on the principles of quantum mechanics [1, 2]. Each interaction has a corresponding gauge symmetry, and the gauge field serves as the mediator of the interaction. Oddly enough, gravity stands out as the lone exception. In Einstein's theory of General Relativity (GR), gravity is described in a purely geometrical way [3, 4]. During the past few centuries, unification has become the central theme for physics. Because of this, one might have wished for a unified framework to describe both gravity and other fundamental interactions. However, gravity may just be too different.

In the geometrical description, gravity does not have a corresponding gauge symmetry or gauge field. In GR, the term *gravity* can be best understood as describing how matter and the spacetime geometry are affecting each other. So it is extremely difficult to try to unify gravity with other fundamental interactions by coming up with a bigger gauge group. In reverse, internal gauge groups can arise through the Kaluza-Klein reduction of extra spacetime dimensions (see [5], and refs there in). In this case, the gauge groups are completely determined by the geometry of the internal spacetime. A unification might be possible if one is content with relating Newton's constant to the strength of other fundamental interactions.

On the other hand, since quantum theories have proved to be the foundation in

This dissertation follows the style of Nuclear Physics B.

describing all the fundamental interactions in particle physics, it is natural to expect a quantum theory of gravity. However, due to the smallness of Newton's constant, the effect of quantum gravity (if it does exist) will be very difficult to detect. The discovery of the black hole thermodynamics [6, 7, 8] in the 1970's opened up a test ground where quantum gravity effects could be important. Another place to look is near the big bang singularity at the beginning of the Universe. Potentially appreciable quantum gravity effects in real experiments have also been proposed (see, e.g. [9]).

During the past few decades, tremendous amount of effort has been put into quantizing gravity and unifying it with the other forces, with only limited success [10, 11, 12]. To date, the discovery of the gauge/gravity duality is regarded as the most significant achievement in the process [13, 14, 15, 16].

In this dissertation, we will be focusing on issues related to black holes.

Black holes are important for many different reasons. First, Einstein's field equations are highly nonlinear, so to get an analytical and physically meaningful solution is a challenging and interesting problem by itself. Second, compelling evidence has been found indicating that black holes may exist in the real world [17]. So black holes can be real objects that worth a lot of attention. Last, but not least, in the study of quantum gravity theories, it is important to have exact solutions so that various ideas can be tested. Even at the classical level, the Schwarzschild solution [18] has played a crucial role in experimentally testing General Relativity [19]. Two major events took place in the middle part of the 1990's. One is that Strominger and Vafa calculated the entropy of a certain type of black holes [20] in 1996 by counting the quantum states. The other is that Maldacena discovered the AdS/CFT correspondence in 1997 [13]. Such development has made it even more important to have more exact solutions, with which one can test the various calculation techniques developed and the various duality relations proposed.

The purpose of this dissertation is to discuss the construction of new black hole solutions and the calculation of the black hole entropy. In Chapter II, we shall report some new black hole solutions that we have found during the past few years [21, 22, 23] and we will discuss some of their properties. In Chapter III, we shall calculate the entropy [24, 25] of various black hole solutions by using the recently proposed Kerr/CFT correspondence [26], which is another example of the gauge/gravity duality. We will demonstrate the applicability of the Kerr/CFT correspondence to all known extremal stationary and axisymmetric black holes. This will be done with the help of two ansatz that can cover all known extremal stationary and axisymmetric black hole solutions.

In the remainder of this introductory chapter, we set out to remind ourselves about how physics has evolved into what it is today, just to get a better idea of what to expect for the future. Especially, we will focus on how unification is brought into the central stage of physics, and we will explain in more detail about how gravity is different from the other fundamental interactions.¹ We will also recapitulate some of the major effort in searching for quantum gravity, focusing on the role played by black holes.

A. The Legacy of Unification

Given the full complexity of the world around us, it is not without a surprise that any sense can be made out of it at all. During the past centuries, significant knowledge has been accumulated about all levels of nature. With it, there comes the even more

¹These will constitute the first two sections of the current chapter. The materials involved are more or less well known to people in different fields in a scattered way. Here I try to put a brief story together in the hope to get a fuller picture. For this reason, references are not provided in general. The trusted website <http://en.wikipedia.org> has been an important source of information.

profound revelation, that all the complexity appears to be governed by only a handful of basic laws. Unification has now become the central theme for physics.

1. The Above

At the cosmic level, the complexity of nature manifests itself in the many different types of objects in the sky. Ptolemy long ago was able to come up with a very complicated system that predicted (then observable) celestial motions to a good accuracy. Although he made the wrong assumption that Earth was at the center of the Universe, his theory was able to withstand challenge for more than a millennium, partially due to the help from the Catholic church.

In the sixteenth and seventeenth centuries a correct picture was drawn for the solar system and Earth was moved to its right position, mainly thanks to the work of Nicolaus Copernicus, Johannes Kepler and Galileo Galilei. Copernicus presented a predictive mathematical model of a heliocentric system in 1543, which gained a substantial support when Galileo found the four largest satellites of Jupiter in 1610 (This essentially falsified the idea that all astronomical objects revolve around the Earth). Kepler published his three laws of planetary motion in 1609 and 1618, based on the observations made by Tycho Brahe. Kepler's three laws provided the first quantitative relations obeyed by the planetary motion. However, Kepler's idea of elliptical planetary orbits was not welcomed by Galileo, because Galileo viewed the shape of an elliptical orbit as less perfect than that of the circular ones. This is an interesting example of how a person's preference can affect his judgement, even for those who are great.

Kepler's laws were finally explained in 1687 by Isaac Newton. His work on the universal gravitation shows that planets stay in orbits and apples fall to the ground due to the same reason: the gravity. Newton gave an explicit mathematical expression

for the gravitational force between any two objects,

$$F = \frac{G_N m_1 m_2}{r^2}, \quad (1.1)$$

where G_N is Newton's constant, and m_1 and m_2 are the gravitational masses of the two objects involved. Newton's theory was firmly established by the discovery of Neptune in 1846 based on an indirect evidence from the orbit irregularities of Uranus. Before that, Friedrich Bessel had also used Newton's theory to predict the existence of a then unknown companion of Sirius based on the changes in Sirius's proper motion. The discovery of Sirius B in 1862 has hence proved the applicability of Newton's theory beyond the solar system.

Although there are problems with Newton's theory of universal gravitation, we will be content for the moment, knowing that the Newtonian gravity is still the best choice when one is dealing with a low energy and slow motion process.

Apart from those related to gravity, another important category of macroscopic phenomena are ones related to electrics, magnetics and light. It was James Clerk Maxwell who showed that all these phenomena are governed by a single theory: the electromagnetism. Maxwell unified electrics and magnetics in the 1860's and 70's. He also suggested that light is the electromagnetic wave, based on the fact that the calculated speed of the electromagnetic wave is closed to the measured speed of light. Maxwell's calculation of the speed of light did not use any particular coordinate system, which means the speed of light is the same for any observer. This result was used by Einstein to propose his theory of Special Relativity in 1905, which established the revolutionary idea that space and time are not independent of each other absolutely. Rather, space and time are related in the process of the Lorentz coordinate transformation, which was a symmetry of Maxwell's equations first noticed by Hendrik Lorentz. The Lorentz transformation goes back to the more intuitive Galileo

transformation in the low speed limit.

2. The Middle

Statistics and thermodynamics serve as the bridge connecting the macroscopic and the microscopic world.

The story of thermodynamics usually starts with the ancient belief that the vacuum is forbidden by nature. This was shown to be inaccurate by Evangelista Torricelli in 1643 using his mercury column experiment. In the experiment Torricelli also demonstrated the effect of the pressure due to the atmosphere. Another important notion to be clarified concerned heat. At the beginning even the difference between hot and cold was a source for puzzle, and there was confusion about the relation between heat and combustion, and people also believed that heat is some substance that cannot be destroyed or produced. So it was a nontrivial progress when it was finally shown that heat is convertible with the work done in a mechanical or electric process. This result was established mainly due to the effort of James Prescott Joule at around the 1840's. His work lead to the discovery of the conservation of energy and the establishment of the first law of thermodynamics. For the other first few laws of thermodynamics, the second law can be traced as far back as to Sadi Carnot in 1824, and the zeroth and the third laws were well established by the early part of the twentieth century.

On the statistics side, Daniel Bernoulli argued in 1738 that gas is made of huge amount of tiny particles and the kinetic motion of the particles is responsible for the pressure and the heat of the gas. This could have been the beginning to use statistics to study thermodynamics. But Bernoulli's insight was not well appreciated by his peers. It was not until 1859, when the kinetic theory started to be developed, that a statistical law in physics describing a velocity distribution of particles was first

formulated by James Clerk Maxwell. Maxwell's work was then generalized by Ludwig Boltzmann, who also gave a statistical explanation of the entropy by relating it to the number of degenerate configurations that are allowed by a classical system.

The most important consequence of the development of the statistical thermodynamics was the discovery of the quantization of energy. At the time when Newton proposed his theory of the universal gravitation, he also put forward three laws describing the relation between the force and the motion. These three laws laid the foundation for the classical mechanics. So by the end of the nineteenth century, it appeared that almost all the observed phenomena had found their explanation. However, some problems did remain. One such problem was related to the black body radiation — the radiation from a closed cavity of constant temperature. The black body radiation displays a continuous spectrum with very particular features, but a satisfactory explanation of which was not available. It was not until 1901 that Max Planck finally provided a statistical answer. And to get a physical explanation for his solution, Planck realized that the energy of the radiation in the cavity must be quantized. This idea of photon quanta was then used by Einstein to successfully explain the photoelectric effect in 1905. These two events marked the beginning of the quantum era.

3. The Beneath

The quest to the microscopic world started with the effort to classify substance found in nature. In this direction the story usually starts with Antoine Lavoisier, who published a list of thirty three elements in 1789. The idea of atoms and molecules in the modern sense was developed and improved by John Dalton and Amedeo Avogadro. A significant progress came when Dmitri Ivanovich Mendeleev and Julius Lothar Meyer published their periodic tables in 1869 and 1870. The direct evidence of the existence

of atoms and molecules was first noticed in 1827 by Robert Brown through the so called Brownian motion of pollen particles in water. However the connection was not made until 1905 when Einstein reasoned that the Brownian motion is due to the water molecules colliding with the pollen particles.

The first elementary particle to be discovered was the electron, by Joseph John Thomson in 1897. Thomson also came up with a plum pudding model for the atoms, which was disproved by his student Ernest Rutherford when Rutherford discovered the atomic nucleus in 1911. Rutherford then came up with the planetary model for atoms, but his model faced a potentially disastrous problem because the electrons are in a constant acceleration around the nucleus, and which means the electrons should radiate and lose energy very quickly. To solve this problem and to explain the spectra lines from the hydrogen atom, Niels Bohr incorporated the idea of quantized orbits into the planetary model in 1913. Bohr's model was then generalized by Arnold Sommerfeld in 1916 by adding the elliptical orbits. Both models met with limited success. Further development was brought forward by Louis de Broglie in 1924 when he generalized Planck and Einstein's light quanta idea to propose the theory of wave-particle duality. This work inspired Erwin Schrödinger to use waves to describe electrons in an atom in 1926. Schrödinger's theory was shown to be equivalent to the matrix formulation of quantum mechanics developed by Werner Heisenberg in 1925. Heisenberg also formulated the *uncertainty principle*. The physical meaning of Schrödinger's wavefunction was clarified by Max Born, who pointed out that the wavefunction is related to the probability of particle distribution. A correct picture of the atoms was then established during the first thirty years of the twentieth century.

In the mean time it was found that the nucleus are composed of protons and neutrons. More and more particles were discovered afterwards, and a large amount of hadrons were found in the 1950's. The idea of quarks was introduced in the process

of classifying the numerous particles found. Today all the hadrons are believed to be made of only six types of quarks. Each quark carries three colors. The need to keep protons and neutrons inside the nucleus lead to the introduction of the strong force, which was then theorized by gauging the color degrees of freedom for quarks in the 1970's. At about the same time the weak interaction was also introduced and was unified with the electromagnetism by Abdus Salam, Sheldon Glashow and Steven Weinberg. To this end, the Standard Model of particle physics took its shape. The interactions are described by the $SU(3) \times SU_L(2) \times U(1)$ gauge symmetries. There are limited number of particles which are deemed as elementary. They include six leptons and their antiparticles, six quarks (each carries three colors) and their antiparticles, twelve gauge fields, and possibly the Higgs fields.

The Standard Model requires three distinct coupling constants for the three symmetry groups. It was shown in the 1970's that all these coupling constants come close to converge at an energy close to the Planck scale [27]. This provided a strong support for grand unified theories proposed to unify the gauge symmetries involved in the Standard Model [28, 29, 30, 31, 32]. Later research has also raised the possibility of the supersymmetrized version of the Standard Model, which predicts numerous superpartners for the particles already known. The existence of the Higgs field and the superparticles is awaiting test at the recently built Large Hadron Collider (LHC) at CERN.

B. The Peculiarity about Gravity

After centuries of development the fundamental physic now has at its foundation several well organized structures, with symmetries playing role of the basic organizing principles. In the case of gravity the requirement of the general coordinate invariance

has lead Einstein to formulate his theory of General Relativity in a general covariant way. In particle physics the structure of the Standard Model is determined both by the Poincaré symmetry of the Minkowski spacetime and by the internal gauge symmetries. The idea of symmetries is well blended with the principle of quantum mechanics in describing the microscopic world. What's more, although statistics and thermodynamics seem to be unrelated to the afore mentioned theories in any fundamental way, they provide the necessary connection between the macroscopic and microscopic world. Now the biggest questions remaining are how the quantum theory is playing a role in gravity, and how gravity is related to other fundamental interactions.

In this section we shall remind ourselves about a significant difference between gravity and other fundamental interactions, which is that gravity in GR is purely geometry. In emphasizing on this difference between gravity and other fundamental interactions, we wish to get a better idea of what to expect (or not to expect) from a quantum theory of gravity.

According to Newton's first and second laws, the velocity of a classical object can be changed only if there is an external force \mathbf{F} , and the acceleration is given by

$$\mathbf{a} = \frac{\mathbf{F}}{m_i} . \quad (1.2)$$

Here m_i is the inertial mass of the object, as compared to the gravitational masses used in (1.1). There is a problem with (1.2) due to the ambiguity in defining the acceleration. Intuitively, when two observers want to compare their results, they use the Galileo transformation for the coordinates,

$$t' = t, \quad \mathbf{x}' = \mathbf{x} + \mathbf{v}t, \quad (1.3)$$

where \mathbf{v} is the velocity of the unprimed observer as seen by the primed one. In the

case when \mathbf{v} is a constant, one gets $\mathbf{a}' = \mathbf{a}$, so both observers will get the same result for the force, which is consistent with (1.2). However, if \mathbf{v} is not a constant, then at least one of the observers will find that his result contradicts (1.2) because $\mathbf{a}' \neq \mathbf{a}$ in general. To solve the problem, some observers need to be singled out as being in the inertial coordinate systems — ones in which (1.2) is satisfied. Then for the coordinate systems accelerating with respect to the inertial ones, one can introduce the inertial force to account for the discrepancy between (1.2) and the observation. The inertial force is a result of the conservation of the momentum, but is not a real force. This can be seen in an inertial coordinate system, where one can see that the inertial force is what's causing the “anti-force” involved in Newton's third law.

In the process of defining the inertial force, some of the coordinate systems are treated as more special than others. This makes sense only if the whole Universe is taken into consideration — an idea often referred to as Mach's principle. Mach's principle is difficult to apply in the Newtonian mechanics, but Einstein's theory of General Relativity gives a concrete example of how Mach's principle can be realized. According to Einstein the space and time can be treated equally, and the spacetime manifold is curved because of the matter distribution of the whole Universe. Similar to Newton's first law, Einstein assumes that the natural motion for a classical object is to follow the geodesics. Acceleration is then defined as any deviation from the geodesic motion. As a result, gravity is nothing but an inertial force due to the deviations from the geodesic motion. For example, the natural motion for a person is to fall to the center of the Earth, but he is prevented from doing so by the support from the ground. Hence, a person standing on the ground is in constant acceleration, and because of this, he feels the gravity — the inertial force.

So gravity in General Relativity is described in a pure geometric way, and there does not need to be a corresponding gauge symmetry nor a gauge field serving as

the force carrier. In contrast, for the three fundamental interactions in the Standard Model, the interactions are purely due to the exchange of the gauge fields. Because of this huge difference, the most direct hope for unification, i.e. to come up with a big gauge group that can cover both gravity and other fundamental interactions, will probably turn out to be in vein. However, unification in the sense of relating Newton's constant to the coupling constants of other fundamental interactions could still be desirable.

On the other hand, although gravity does not have a force carrier in the usual sense, there are still dynamical degrees of freedom related to the metric. Indirect evidence for the existence of the graviton in the form of the gravitational wave has been found in the 1970's [33]. Direct detection of the gravitational wave is currently under the way [34].

Finally, there is always the hope for better alternatives to GR. In GR, the identification of gravity as the inertial force is based on the equivalence principle, part of which says that the gravitational mass (as in (1.1)) of an object is the same as its inertial mass (as in (1.2)). If the equivalence principle fails, then the pure geometrical description of gravity will also fail, and then one may need to treat gravity as a usual force mediated by some force carrier.

C. Quantum Gravity and Black Holes

The study of quantum gravity started as early as the 1930's. Several major formalisms have been developed over the years. A concise history of the search for quantum gravity can be found in [10], and a detailed discussion of various formalisms used to quantize gravity can be found in [11].

The most straightforward choice is the covariant formalism [35, 36, 37, 38], in

which the quantization is done by perturbing the metric $g_{\mu\nu}$ around a chosen background $\bar{g}_{\mu\nu}$, $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. It was found in the 1970's that pure Einstein's gravity is nonrenormalizable [39, 40]. Efforts in this direction are still being made.

A second method is the Hamiltonian formalism (the Canonical formalism) by using the Dirac quantization [41]. In this method, one first follows the Arnowitt-Deser-Misner (ADM) approach [42] to write the metric as

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \quad (1.4)$$

The spatial metric g_{ij} and its conjugate momentum $\pi^{ij} = \partial\mathcal{L}/\partial\dot{g}_{ij}$ are treated as the canonical variables of the system, while the lapse N and shift functions N^i turn out to be Lagrange multipliers in the action [11],

$$I = \int d^4x \left(\pi^{ij} \partial_t g_{ij} - N\mathcal{H} - N_i \mathcal{H}^i \right) + \text{surface terms}. \quad (1.5)$$

Corresponding to N and N^i , the Hamiltonian and momentum constraints are

$$\mathcal{H} = \frac{16\pi G}{\sqrt{g}} \left(\pi_{ij} \pi^{ij} - \frac{1}{2} \pi^2 \right) - \frac{\sqrt{g}}{16\pi G} \left[{}^{(3)}R - 2\Lambda \right], \quad (1.6)$$

$$\mathcal{H}^i = -2 {}^{(3)}\nabla_j \pi^{ij}. \quad (1.7)$$

Upon quantization, $\pi^{ij} = -i\delta/\delta g_{ij}$. The momentum constraints require that the wave functional must be diffeomorphism invariant, and the Hamiltonian constraint leads to the Wheeler-DeWitt equation. Due to the difficulty with the Wheeler-DeWitt equation, the Hamiltonian constraint cannot be solved in general. However, it was later discovered that the constraints can be made simpler by using the Ashtekar variables [43, 44]. The Ashtekar variables lead to constraints that can be solved by using the Wilson loop. This laid down the foundation for the development of the Loop Quantum Gravity [45].

The last formalism to be mentioned is the string theory [46], which is currently the most widely accepted candidate for quantum gravity. In the theory, the fundamental dynamical degrees of freedom are related to the vibration of some fundamental strings. Consistency of the theory requires the spacetime to be 26 or 10 dimensions. So to make connections with the real world, people need to explain why the extra dimensions are not observed. One of the most important ideas developed in string theory is duality [47]. And the most important discovery is the duality between quantum gravity and some field theories [13, 14, 15, 16]. A comparison of various aspects of string theory and loop quantum gravity can be found in [12].

Given a candidate theory for quantum gravity, it is important to have some kind of test to see if the theory describes the real world. However, even with advancements of modern technologies, the expected quantum gravity effect is still very difficult to detect [9]. So it is understandable that at the current stage, the less direct evidence still plays a significant role. Among such evidence, the most important is the capability to explain the black hole entropy.

The first black hole solution was found by Karl Schwarzschild [18], soon after Einstein published his field equations for General Relativity. The idea that a black hole has an entropy proportional to its horizon area was first proposed by Jacob D. Bekenstein in 1973 [6], based on the similarity between the second law of thermodynamics and the fact that the horizon area of black holes tends to increase in any natural process [48]. Results analogous to the four laws of thermodynamics were also developed [7]. Compelling evidence was then provided by Hawking when he discovered that black holes radiate like black bodies with finite temperatures [8]. Hawking found that the temperature and the entropy of a black hole are given by

$$T = \frac{\kappa}{2\pi}, \quad S = \frac{\mathcal{A}_{rea}}{4}, \quad (1.8)$$

where κ is the surface gravity and \mathcal{A}_{rea} is the area of the horizon. The entropy is usually understood to be related to the number of the microscopic states making up a system. Given the successful story that the study of the black body radiation has lead to the discovery of the quantum theory at the beginning of the 20'th century, people hope that the study of the black hole entropy may shed some light on the search for quantum gravity.

The peculiar feature that the entropy of a black hole is proportional to the area of the horizon has an important implication, that the information about quantum gravity may be completely contained in the boundary of where the theory is defined. This idea, dubbed the holographic principle, was first proposed by Gerard t' Hooft and others [49, 50]. In 1997, Maldacena conjectured that String theory on the $S^5 \times AdS_5$ background is dual to the large N limit of the $\mathcal{N} = 4$ super Yang-Mill theory [13]. This provided a concrete realization of the holographic principle. The success inspired the effort to use a dual field theory to define quantum gravity. Because the duality relation necessarily involves one weakly interacting and one strongly interacting theories on the opposite sides, it also makes it possible to use gravity to study the strongly coupled problems in field theories, such as those in the quantum chromodynamics (QCD). In all these calculations, it is essential to have exact solutions on the gravity side.

The first successful calculation of the black hole entropy (by counting the microscopic states) was done in 1996 by Strominger and Vafa [20] in String theory. Soon a calculation was also done in Loop Quantum Gravity which reached the result that the entropy is proportional to the area [51]. Over the years, several more methods have been developed to calculate the entropy (for refs, see [52]).

For the Strominger and Vafa calculation, it was later realized that neither string theory nor supersymmetry was crucial in the calculation of the entropy [53]. What's important are possible conformal symmetries related to the black hole horizon [54,

55, 56, 57, 59, 60, 52], defined with appropriate boundary conditions. The existence of conformal symmetries is related to the existence of a conformal field theory. In the spirit of the AdS/CFT correspondence, one can then use the conformal field theory to define the quantum gravity theory which is unknown otherwise. The entropy can be found from the central charge of the Virasoro algebra that is involved, with the help of Cardy's formula [61]. So at the end of the day, to calculate the entropy turns out to be discussing the dynamics on the horizon of a black hole. This is reasonable because, according to Boltzmann, the entropy is related to the number of degenerate states for a given classical system. For a black hole, one should then look for the degeneracies on the horizon, because any changes outside of the horizon will lead to a different black hole, while changes inside the horizon are assumed to be cut off from the outside world.

An early effort in calculating the black hole entropy by using the horizon dynamics was carried out in [62]. Throughout the years, a lot of other methods have also been proposed [52]. The latest development along this line has been the proposal of the Kerr/CFT correspondence by Strominger and collaborators [26]. There is a significant difference from the previous efforts [52], however, in that [26] does not use the diffeomorphism of the horizon directly, but rather the symmetries defined at the spatial infinity of the near-horizon metric. The calculation has been shown to work for all the cases being checked (for refs., see [63]). In this dissertation, we will prove the applicability of the Kerr/CFT correspondence to all known extremal stationary and axisymmetric black holes. However, notice that there are limitations to the current understanding of the Kerr/CFT correspondence. Firstly, the method only works for extremal and near extremal black holes [63, 64]. Secondly, although the method is powerful enough to give a correct counting of the entropy for all known extremal stationary and axisymmetric black holes, it tells very little about the under-

lying quantum gravity theory. At the current stage, one can identify nothing more than the symmetry group and the central charge of the dual conformal field theory.

CHAPTER II

BLACK HOLE SOLUTIONS*

Apart from the fact that looking for an exact solution to Einstein's equations is itself a challenging and interesting problem, it is also crucial to have exact solutions available so that various ideas can be tested when looking for the quantum gravity theory.

In the search for new solutions, there can be solution generating techniques in some cases. For example, a global symmetry combined with the Kaluza-Klein reduction process can be used to add new charge parameters to a known solution. For a lot of other cases, however, a solution generating technique is not available and one has to use the less delicate guess-and-trial method. For such cases, a good ansatz for the metric and for the matter fields is of crucial importance. A good guess of the right ansatz can often be learned from existing solutions. In this chapter, we present two examples found in this way. One example is a Plebanski-Demianski type solution in five dimensions [22, 23], and the other is a solution to the five dimensional $\mathcal{N} = 2$ supergravity coupled to two vector multiplets [21]. The work was done with H. Lü and C. N. Pope in [22, 23] and C. N. Pope in [21].

The first three sections of this chapter serve as an introduction. In section A we briefly explain some basic features about black holes and explain why we want

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to study black hole solutions. In sections B and C we introduce some of the most important black hole solutions that are known in various dimensions and which are also most relevant to our new solutions. We discuss our result in sections D and E.

A. Why Care about Black Holes?

Einstein published his field equations of General Relativity (GR) in 1915,

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = T_{\mu\nu}. \quad (2.1)$$

These equations can be derived from the Einstein-Hilbert action

$$S = \int d^4x \sqrt{-g} (R + \mathcal{L}_M). \quad (2.2)$$

The theory won immediate recognition, partially thanks to the fact that its prediction of the deflection of star light by Sun was confirmed in Eddington's 1915 expedition. Contrary to this, the importance of the solutions to Einstein's equations was not appreciated until much later.

The first exact solution to (2.1) was found by Schwarzschild [18], soon after Einstein published his field equations. The solution describes a black hole with a point mass m located at the origin. So the region outside the source is empty and the (2.1) reduces to

$$R_{\mu\nu} = 0. \quad (2.3)$$

The Schwarzschild solution is given by

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (2.4)$$

$$f = 1 - \frac{2m}{r}. \quad (2.5)$$

The name “black hole” was made known by John Wheeler in 1967.

The metric (2.4) approaches that of a Minkowski spacetime when $r \rightarrow +\infty$ as expected, where the effect of a localized mass is negligible. On the other hand, the metric is singular at $r = 2m$. What's worse, there is a genuine curvature singularity at $r = 0$, as can be seen from

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48m^2}{r^6}. \quad (2.6)$$

Physical entities with such bizarre features were not expected to exist in the real world, and so the physical significance of the Schwarzschild solution was not well appreciated at the time when the solution was found. Not much change of attitude would take place in the dozens of years that followed [65]. The first proper understanding of the apparent singularity at $r = 2m$ was due to D. Finkelstein, who showed that the singularity is due to a bad choice of the coordinate system, and that the $r = 2m$ surface in fact acts as “a perfect unidirectional membrane” [66]. Soon after, Kruskal and Szekeres proposed a new coordinate system which completely removes this coordinate singularity [67, 68]. A better picture of the causal structure of the solution was later achieved with the Penrose diagram. The $r = 2m$ surface is now called the horizon of the Schwarzschild black hole.

For the genuine curvature singularity at $r = 0$, it cannot be removed by a coordinate transformation. In Penrose's cosmic censorship hypothesis, it is believed that such singularities will not arise in any natural process and so causality is always preserved. The only exception allowed might be the cosmic singularity at the beginning of the Universe in the Big Bang theory.

Despite the development in understanding the Schwarzschild solution itself, one can still ask if a black hole really exists in nature. In this respect, the first progress was made in the early 1930's when Chandrasekhar discovered an upper limit for the mass of a completely degenerate configuration [69]. The Chandrasekhar limit for the white

dwarf is $1.4M_{\odot}$ [65]. In 1939, Oppenheimer and Volkoff demonstrated the formation of a black hole from the collapse of a homogenous sphere of pressureless gas [70]. In the case studied by them, the electron degeneracy pressure already fails, and the effect of GR should be taken into account. Now it is known that a cold star with mass greater than about $3M_{\odot}$ will eventually collapse and form a black hole [65]. So from the theory side, there is intriguing evidence that black holes might exist in nature. Compelling experimental evidence has also been accumulated through astronomical observations [17].

From the modern perspective, since the discovery of Bekenstein-Hawking entropy [6, 7], all candidates of the quantum gravity theory are challenged to come up with a microscopic explanation of the black hole entropy. What's more, the development of string theory has made it desirable to have solutions in dimensions other than four [71]. Some success has been made in calculating the entropy in string theory and in Loop Quantum Gravity in 1996 and 1997 [20, 51]. Also in 1997, Maldacena made his important discovery of the AdS/CFT correspondence [13]. Such development has inspired even more interest in studying the black hole entropy, and has made it even more important to have more exact solutions so that various ideas can be tested.

B. Black Hole Solutions in General Relativity

Since invaluable lessons can be learned from the structure of existing solutions, especially when one is trying to construct new ones, we shall write out most of the solutions that we mention in this section.

1. Solutions in Four Dimensions

Soon after Schwarzschild, H. Reissner and G. Nordström generalized the solution to include an electric charge [72, 73]. In this case, the matter contribution to (2.2) is

$$\mathcal{L}_M = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (2.7)$$

Equations of motion derived from this action are

$$R_{\mu\nu} = \frac{1}{2}F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{8}F^2g_{\mu\nu}, \quad \partial_\mu(\sqrt{-g}F^{\mu\nu}) = 0. \quad (2.8)$$

The Reissner-Nordström solution is given by

$$\begin{aligned} ds^2 &= -f dt^2 + \frac{dr^2}{f} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \\ A &= \frac{2q}{r} dt, \quad f = 1 - \frac{2m}{r} + \frac{q^2}{r^2}. \end{aligned} \quad (2.9)$$

Comparing this to the Schwarzschild solution in (2.4), the only difference is in the function f and in the inclusion of a $U(1)$ gauge field. There are uniqueness theorems (see, e.g. [74]) which establish that black holes in four dimensions are completely characterized by their global charges, such as mass, electric charge and an angular momentum. The angular momentum was not introduced until 1963, when Kerr found a solution describing a neutral but rotating black hole [75]. The solution is given by

$$\begin{aligned} ds^2 &= \rho^2 \left(\frac{dr^2}{\Delta} + d\theta^2 \right) + \frac{\sin^2\theta}{\rho^2} \left[a dt - (r^2 + a^2) d\phi \right]^2 \\ &\quad - \frac{\Delta}{\rho^2} (dt - a \sin^2\theta d\phi)^2, \\ \Delta &= r^2 + a^2 - 2mr, \quad \rho^2 = r^2 + a^2 \cos^2\theta. \end{aligned} \quad (2.10)$$

The angular momentum of the Kerr black hole is given by $J = ma$. The charged version of the solution was found two years later by Newman [76], where apart from

the addition of a $U(1)$ gauge field the only change is in the Δ function,

$$A = \frac{2q_e r}{\rho^2}(dt - a \sin^2 \theta d\phi), \quad \Delta = r^2 + a^2 - 2mr + q_e^2. \quad (2.11)$$

Notice that (2.10) can be put into the Kerr-Schild form [77],

$$\begin{aligned} ds^2 &= -d\tilde{t} + dx^2 + dy^2 + dz^2 + \frac{2m}{r + a^2 z^2 / r^3} K^2, \\ K &= d\tilde{t} + \frac{r(xdx + ydy) + a(xdy - ydx)}{r^2 + a^2} + \frac{zdz}{r}, \end{aligned} \quad (2.12)$$

by using the following coordinate transformations

$$\begin{aligned} x &= \sqrt{r^2 + a^2} \sin \theta \cos \tilde{\phi}, \quad y = \sqrt{r^2 + a^2} \sin \theta \sin \tilde{\phi}, \quad z = r \cos \theta, \\ d\tilde{t} &= dt + \frac{2mr}{\Delta} dr, \quad d\tilde{\phi} = -d\phi - \frac{2amr}{\Delta(r^2 + a^2)} dr. \end{aligned} \quad (2.13)$$

In the Kerr-Schild form (2.12), the Kerr solution appears to be a linear perturbation over the flat Minkowski background. The one form K is null both under the background metric and under the full metric. The Kerr-Schild form can be very helpful when search for Kerr like solutions in higher dimensions (see, e.g. [71, 78]).

In the case when there is a cosmological constant, Einstein's equations become

$$R_{\mu\nu} = \Lambda g_{\mu\nu}. \quad (2.14)$$

The simplest solution is the de Sitter (dS) space found in 1916 \sim 1917, which describes an otherwise empty spacetime with a positive cosmological constant. The solution is given by

$$\begin{aligned} ds^2 &= -f dt^2 + \frac{dr^2}{f} + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \\ f &= 1 - g^2 r^2, \quad \Lambda = 3g^2. \end{aligned} \quad (2.15)$$

There is a cosmological horizon for any observer inside the dS space. With a point

mass, a Schwarzschild-dS black hole can be generated,

$$\begin{aligned} ds^2 &= -f dt^2 + \frac{dr^2}{f} + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \\ f &= 1 - \frac{2m}{r} - g^2 r^2, \quad \Lambda = 3g^2. \end{aligned} \quad (2.16)$$

It has a black hole horizon and a cosmological horizon, and the former can be no bigger than the latter. In the case when the black hole horizon approaches the cosmological horizon in size, one can no longer distinguish between the two, and this is called the Nariai limit [79].

When a $U(1)$ gauge field is also included, one gets the Einstein-Maxwell system with a cosmological constant,

$$R_{\mu\nu} = \frac{1}{2}F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{8}F^2 g_{\mu\nu} + \Lambda g_{\mu\nu}, \quad \partial_{\mu}(\sqrt{-g}F^{\mu\nu}) = 0. \quad (2.17)$$

A fairly general solution called the Kerr-Newman-AdS solution, characterized by a mass, a rotation, an electric charge and a negative cosmological constant $\Lambda = -3g^2$, is found to be

$$\begin{aligned} ds^2 &= \rho^2 \left(\frac{dr^2}{\Delta} + \frac{d\theta^2}{\Delta_{\theta}} \right) + \frac{\Delta_{\theta} \sin^2 \theta}{\rho^2} \left[a dt - \frac{r^2 + a^2}{\Xi} d\phi \right]^2 \\ &\quad - \frac{\Delta}{\rho^2} \left(dt - \frac{a \sin^2 \theta}{\Xi} d\phi \right)^2, \\ A &= \frac{2q_e r}{\rho^2} \left(dt - \frac{a \sin^2 \theta}{\Xi} d\phi \right), \\ \Delta &= (r^2 + a^2)(1 + g^2 r^2) - 2mr + q_e^2, \quad \Xi = 1 - g^2 a^2, \\ \rho^2 &= r^2 + a^2 \cos^2 \theta, \quad \Delta_{\theta} = 1 - g^2 a^2 \cos^2 \theta. \end{aligned} \quad (2.18)$$

The solution describes a charged rotating black hole sitting in an AdS back ground. One can recover solutions in (2.9), (2.10) and (2.18) by turning off the corresponding parameter.

Apart from these, another important class of solutions are ones with a NUT parameter. Just like a magnetic charge is the dual of an electric charge, the NUT parameter can be viewed as the gravitational analog of the magnetic charge, dual to the mass. The first such example was found by Taub (1951), Newman, Tamubrino and Unti (1963) [80, 81]. The solution is given by

$$\begin{aligned} ds^2 &= -f \left[dt - 4n \sin^2 \left(\frac{\theta}{2} \right) d\phi \right]^2 + \frac{dr^2}{f} + (r^2 + n^2)(d\theta^2 + \sin^2 \theta d\phi^2), \\ f &= 1 - \frac{2(mr + n^2)}{r^2 + n^2}, \end{aligned} \quad (2.19)$$

which goes back to the Schwarzschild solution (2.4) when the NUT parameter is zero, $n = 0$. When the rotation is added, one gets the so called Kerr-NUT solution by Demianski and Newman [82],

$$\begin{aligned} ds^2 &= \rho^2 \left(\frac{dr^2}{\Delta} + d\theta^2 \right) + \frac{\sin^2 \theta}{\rho^2} \left[a dt - (r^2 + a^2 + n^2) d\phi \right]^2 \\ &\quad - \frac{\Delta}{\rho^2} \left[dt - (a \sin^2 \theta + 2n \cos \theta) d\phi \right]^2, \\ \Delta &= r^2 + a^2 - 2mr - n^2, \quad \rho^2 = r^2 + (a \cos \theta - n)^2. \end{aligned} \quad (2.20)$$

Again, one can set $a = 0$ to recover (2.19). In the case when there is also a cosmological constant $\Lambda = -3g^2$, the Kerr-NUT-AdS solutions was found in [83],

$$\begin{aligned} ds^2 &= \rho^2 \left(\frac{dr^2}{\Delta} + \frac{d\theta^2}{\Delta_\theta} \right) + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left[a dt - \frac{r^2 + a^2 + n^2}{1 - g^2 a^2} d\phi \right]^2 \\ &\quad - \frac{\Delta}{\rho^2} \left[dt - \frac{a \sin^2 \theta + 2n \cos \theta}{1 - g^2 a^2} d\phi \right]^2, \\ \Delta_\theta &= 1 - g^2 a^2 \cos^2 \theta - g^2 n(n - 2a \cos \theta) \left[1 - \frac{(n - a \cos \theta)^2}{a^2 \sin^2 \theta} \right], \\ \Delta &= (r^2 + a^2)(1 + g^2 r^2) - 2mr - n^2, \quad \rho^2 = r^2 + (n - a \cos \theta)^2. \end{aligned} \quad (2.21)$$

The solution goes back to (2.20) when $g = 0$, and to (2.18) when $q_e = n = 0$.

All the solutions mentioned above were unified into a single solution in 1976 by

Plebanski and Demianski [84],

$$\begin{aligned}
ds^2 &= \frac{1}{(1-xy)^2} \left[(x^2+y^2) \left(\frac{dx^2}{X} + \frac{dy^2}{Y} \right) + \frac{Y(d\psi_1 + x^2 d\psi_2)^2}{x^2+y^2} \right. \\
&\quad \left. - \frac{X(d\psi_1 - y^2 d\psi_2)^2}{x^2+y^2} \right], \\
A &= \frac{2(q_1x + q_2y)d\psi_1 + 2xy(q_2x - q_1y)d\psi_2}{x^2+y^2}, \quad Q^2 = q_1^2 + q_2^2, \\
X &= \frac{g^2}{2} + \frac{Q^2}{2} + c_4 - 2Mx + c_2x^2 + 2Lx^3 + \left(\frac{g^2}{2} + \frac{Q^2}{2} - c_4 \right) x^4, \\
Y &= \frac{g^2}{2} - \frac{Q^2}{2} + c_4 - 2Ly - c_2y^2 + 2My^3 + \left(\frac{g^2}{2} - \frac{Q^2}{2} - c_4 \right) y^4. \quad (2.22)
\end{aligned}$$

This solution contains seven parameters, g , M , L , q_1 , q_2 , c_2 and c_4 . To see its relation with (2.18) and (2.21), one can make the following scaling,

$$\begin{aligned}
x &\rightarrow \epsilon x, \quad y \rightarrow \epsilon y, \quad M \rightarrow \epsilon^3 m, \quad L \rightarrow \epsilon^3 n, \\
q_1 &\rightarrow \epsilon^2 q_1, \quad q_2 \rightarrow \epsilon^2 q_2, \quad c_2 \rightarrow \epsilon^2(1 + g^2 a^2), \\
g^2 &\rightarrow g^2 + \epsilon^4(a^2 - n^2), \quad c_4 \rightarrow -\frac{g^2}{2} + \frac{a^2 - n^2 + q_1^2 + q_2^2}{2} \epsilon^4, \\
\psi_1 &\rightarrow \frac{1}{\epsilon} \left(t - \frac{a\phi}{1 - g^2 a^2} \right), \quad \psi_2 \rightarrow -\frac{1}{\epsilon^3} \frac{\phi}{a(1 - g^2 a^2)}. \quad (2.23)
\end{aligned}$$

After sending $\epsilon \rightarrow 0$, one gets the Kerr-Newman-NUT-AdS solution,

$$\begin{aligned}
ds^2 &= (x^2+y^2) \left(\frac{dx^2}{X} + \frac{dy^2}{Y} \right) + \frac{Y}{x^2+y^2} \left[dt - \frac{(a^2+x^2)d\phi}{a(1-g^2a^2)} \right]^2 \\
&\quad - \frac{X}{x^2+y^2} \left[dt - \frac{(a^2-y^2)d\phi}{a(1-g^2a^2)} \right]^2, \\
A &= \frac{2}{x^2+y^2} \left[(q_1x + q_2y) \left(dt - \frac{ad\phi}{1-g^2a^2} \right) - \frac{xy(q_2x - q_1y)d\phi}{a(1-g^2a^2)} \right], \\
X &= (a^2+x^2)(1+g^2x^2) - 2mx - n^2 + q_1^2 + q_2^2, \\
Y &= (a^2-y^2)(1-g^2y^2) - 2ny - n^2. \quad (2.24)
\end{aligned}$$

For this solution, it is easy to tell that g is related to the cosmological constant ($\Lambda = -3g^2$), a is related to the rotation, m is related to the mass, n is the NUT

parameter, q_1 is related to the electric charge, and q_2 is related to the magnetic charge. One recovers (2.21) with $q_1 = q_2 = 0$, and (2.18) with $q_1 = q_e$ and $n = q_2 = 0$. The solution (2.22) contains one more parameter than (2.24), which is related to an acceleration.

Apart from all these, there are many other exact solutions to Einstein's field equations in four dimensions. Some of them do not describe a black hole space time, like pp-wave solutions. Also, some of them have sources other than the electromagnetic field. For example, one could have scalar fields or nonabelian gauge fields. Most of the known solutions are categorized in [85].

2. Solutions in Higher Dimensions

In dimensions $d \geq 4$, the Schwarzschild-like solutions were found in 1963 by Tangherlini [86],

$$\begin{aligned} ds_d^2 &= -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_{d-2}, \\ f &= 1 - \frac{c_m}{r^{d-3}}. \end{aligned} \quad (2.25)$$

Here $d\Omega_{d-2}$ describes the metric of an $d - 2$ dimensional unit sphere. When a $U(1)$ gauge field and a cosmological constant are added, one gets a new solution with the same form of the metric but with

$$\begin{aligned} f &= 1 - \frac{c_m}{r^{d-3}} + \frac{c_q^2}{r^{2(d-3)}} + g^2 r^2, \\ A &= \sqrt{\frac{2(d-2)}{d-3}} \frac{c_q dt}{r^{d-3}}, \end{aligned} \quad (2.26)$$

$$\Lambda = -\frac{(d-1)(d-2)}{2} g^2. \quad (2.27)$$

General rotating black hole solutions in arbitrary higher dimensions were found by Myers and Perry in 1986 [71]. Here we shall follow the notations in [78]. The form of

the solutions differs for even and odd dimensions. In odd dimensions, $d = 2n + 1$,

$$\begin{aligned} ds^2 &= -dt^2 + Fdr^2 + \sum_{i=1}^n (r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2) + \frac{2m}{U}K^2, \\ K &= dt + Fdr - \sum_{i=1}^n a_i \mu_i^2 d\phi_i, \quad U = \prod_{i=1}^n (r^2 + a_i^2) \sum_{j=1}^n \frac{\mu_j^2}{r^2 + a_j^2}. \end{aligned} \quad (2.28)$$

In even dimensions, $d = 2n$,

$$\begin{aligned} ds^2 &= -dt^2 + Fdr^2 + (r^2 + a_n^2)d\mu_n^2 + \sum_{i=1}^{n-1} (r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2) + \frac{2m}{U}K^2, \\ K &= dt + Fdr - \sum_{i=1}^{n-1} a_i \mu_i^2 d\phi_i, \quad U = r \prod_{i=1}^{n-1} (r^2 + a_i^2) \sum_{j=1}^n \frac{\mu_j^2}{r^2 + a_j^2}. \end{aligned} \quad (2.29)$$

For both cases, one has $\sum_{i=1}^n \mu_i^2 = 1$ and

$$F = r^2 \sum_{i=1}^n \frac{\mu_i^2}{r^2 + a_i^2}. \quad (2.30)$$

Note there are n azimuthal angles ϕ_i and so n rotation parameters a_i in odd dimensions, while in even dimensions, there are only $n - 1$ azimuthal angles and $n - 1$ rotations.

The case with a cosmological constant was first found in five dimensions by Hawking, Hunter and Taylor-Robinson [87]. Here we shall follow the notations in [88]. The solution is given by

$$\begin{aligned} ds^2 &= (x^2 + y^2) \left(\frac{dx^2}{X} + \frac{dy^2}{Y} \right) - \frac{X(dt - y^2 d\phi_1)}{x^2 + y^2} + \frac{Y(dt + x^2 d\phi_1)}{x^2 + y^2} \\ &\quad + \frac{a^2 b^2}{x^2 y^2} \left[dt + (x^2 - y^2) d\phi_1 + x^2 y^2 d\phi_2 \right]^2, \\ X &= \frac{(x^2 + a^2)(x^2 + b^2)(1 + g^2 x^2)}{x^2} - 2m, \\ Y &= \frac{(y^2 - a^2)(b^2 - y^2)(1 - g^2 y^2)}{y^2}. \end{aligned} \quad (2.31)$$

Here the cosmological constant is given by (2.27) with $d = 5$. The general Kerr-de

Sitter metrics in arbitrary dimensions are found in 2004 by Gibbons, Lü, Page and Pope [78]. Later NUT parameters are also included, and the general Kerr-NUT-AdS solutions in arbitrary dimensions are found in 2006 by Chen, Lü and Pope [88]. The general Kerr-de Sitter solutions are found with the help of the Kerr-Schild form as in (2.12), (2.28) and (2.29), while the general Kerr-NUT-AdS solutions are found with the help of metrics of the form (2.24) and (2.31), which have proven very helpful in producing new and general solutions.

The horizon of a black hole has the topology of a sphere. This appears to be the only choice for black objects in four dimensions. When one goes to higher dimensions, more choices become available. In 2001, a rotating black ring solution in five dimensions was found by Emparan and Reall [89]. The solution is asymptotically flat and has a horizon of topology $S^2 \times S^1$. The black ring rotates along the S^1 direction, and the rotation prevents the object from collapsing. The configuration of a black ring outside of a black hole, called black Saturn, was found in 2007 by Elvang and Figueras [90]. Other interesting configurations, like two rings rotating perpendicular to each other, called bicycling black rings, have also been found [91].

Up to now, we have only discussed solutions to the pure Einstein gravity or Einstein gravity coupled to the Maxwell field. Black hole solutions in supergravity theories will be discussed in the next section.

As an outcome of studying the solutions discussed in this section, we have found a Plebanski-Demianski type (2.22) solution in five dimensions [22, 23]. We shall discuss the solution in detail in section D.

C. Black Hole Solutions in Supergravity Theories

The importance of black hole solutions in supergravity theories began to be recognized after the first successful calculation of black hole entropy in string theory by Strominger and Vafa in 1996 [20]. Even more weight was added when Maldacena discovered the AdS/CFT correspondence in 1997 [13].

An good example to start with is the solution found by Breckenridge, Myers, Peet and Vafa (BMPV) in 1996 [92],

$$\begin{aligned}
 ds^2 &= - \left(1 - \frac{\mu}{r^2}\right)^2 \left(dt - \frac{\mu w (\sin^2 \theta d\varphi - \cos^2 \theta d\psi)}{r^2 - \mu}\right)^2 \\
 &\quad + \frac{dr^2}{\left(1 - \frac{\mu}{r^2}\right)^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2 + \cos^2 \theta d\psi^2), \\
 A &= \frac{\sqrt{3}\mu (dt + w \sin^2 \theta d\varphi - w \cos^2 \theta d\psi)}{r^2}.
 \end{aligned} \tag{2.32}$$

This solution has two equal rotations and one electric charge. It is a solution to the minimal $d = 5$ supergravity coupled to a vector field,

$$\mathcal{L} = \sqrt{-g} \left(R - \frac{1}{4} F^2 \right) + \frac{1}{12\sqrt{3}} \varepsilon^{\mu\nu\rho\sigma\lambda} F_{\mu\nu} F_{\rho\sigma} A_\lambda. \tag{2.33}$$

The corresponding equations of motion are

$$\begin{aligned}
 R_{\mu\nu} - \left(\frac{1}{2} F_{\mu\alpha} F_\nu{}^\alpha - \frac{1}{12} F^2 g_{\mu\nu} \right) &= 0, \\
 \partial_\mu (\sqrt{-g} F^{\mu\lambda}) + \frac{1}{4\sqrt{3}} \varepsilon^{\mu\nu\rho\sigma\lambda} F_{\mu\nu} F_{\rho\sigma} &= 0.
 \end{aligned} \tag{2.34}$$

The BMPV solution (2.32) is supersymmetric. Supersymmetric black holes are important in that the counting of their microstates can be most reliably done because radiative corrections are suppressed by the supersymmetries that they preserve. Many other examples of supersymmetric black hole solutions can be found in [93, 94] and references therein.

Supersymmetric solutions necessarily have a zero temperature. Non-zero temperature will have to come from the more general non-extremal solutions. It is possible to obtain supersymmetric solutions from non-extremal ones by taking appropriate limits. It is also possible to calculate the entropy for near extremal black hole solutions (see e.g. [95]).

A particularly interesting non-extremal example is the solution found by Cvetič and Youm in 1996 [96]. It is a solution to the so called STU model,

$$\begin{aligned} \mathcal{L} = & \sqrt{|g|} \left[R - \frac{1}{2} \sum_{\alpha=1}^2 (\partial \varphi_\alpha)^2 + \sum_{i=1}^3 \left(4g^2 X_i^{-1} - \frac{1}{4} X_i^{-2} \mathcal{F}_{\mu\nu}^i \mathcal{F}^{i\mu\nu} \right) \right] \\ & + \frac{1}{24} |\varepsilon_{ijk}| \varepsilon^{uv\rho\sigma\lambda} \mathcal{F}_{uv}^i \mathcal{F}_{\rho\sigma}^j \mathcal{A}_\lambda^k, \end{aligned} \quad (2.35)$$

where $|g|$ is the absolute value of the determinant of the metric, which is to be distinguished from the parameter g^2 (which is related to the cosmological constant by $\Lambda = -6g^2$). The quantities X_i are related to two scalar fields φ_1 and φ_2 ,

$$X_1 = e^{-\frac{1}{\sqrt{6}}\varphi_1 - \frac{1}{\sqrt{2}}\varphi_2}, \quad X_2 = e^{-\frac{1}{\sqrt{6}}\varphi_1 + \frac{1}{\sqrt{2}}\varphi_2}, \quad X_3 = e^{\frac{2}{\sqrt{6}}\varphi_1}. \quad (2.36)$$

The equations of motion are

$$\begin{aligned} 0 &= R_{\mu\nu} + \sum_{i=1}^3 \left[\left(\frac{4g^2}{3X_i} + \frac{(F^i)^2}{12X_i^2} \right) g_{\mu\nu} - \frac{F_{a\mu}^i F_{\nu}^{ia}}{2X_i^2} \right] - \partial_\mu \ln X_3 \partial_\nu \ln X_3 \\ &\quad + \frac{\partial_\mu \ln X_1 \partial_\nu \ln X_2 + \partial_\nu \ln X_1 \partial_\mu \ln X_2}{2}, \\ 0 &= \partial_\mu \left(\sqrt{|g|} \frac{F^{i\mu\lambda}}{X_i^2} \right) + \frac{1}{8} |\varepsilon_{ijk}| \varepsilon^{\mu\nu\rho\sigma\lambda} F_{\mu\nu}^j F_{\rho\sigma}^k, \\ 0 &= \nabla_\mu \partial^\mu \ln \left(\frac{X_i}{X_3} \right) - 4g^2 \left(\frac{1}{X_i} - \frac{1}{X_3} \right) + \frac{1}{2} \left[\frac{(F^i)^2}{X_i^2} - \frac{(F^3)^2}{X_3^2} \right]. \end{aligned} \quad (2.37)$$

The Cvetič-Youm solution contains all the parameters but the cosmological constant ($g = 0$), so it is actually a solution to the ungauged theory. The solution is cast into a simpler form in [97]. We shall copy it here because several other solutions to (2.37)

can also be put into a similar form,

$$\begin{aligned}
ds^2 &= (h_1 h_2 h_3)^{1/3} \left[\frac{dx^2}{4X} + \frac{dy^2}{4Y} + \frac{U}{G} \left(d\chi - \frac{Z}{U} d\sigma \right)^2 + \frac{XY}{U} d\sigma^2 \right] \\
&\quad - \frac{G (dt + \mathcal{A})^2}{(h_1 h_2 h_3)^{2/3}}, \\
\mathcal{A} &= \frac{2m c_1 c_2 c_3 [(a^2 + b^2 - y)d\sigma - abd\chi]}{x + y - 2m} - \frac{2m s_1 s_2 s_3 (abd\sigma - yd\chi)}{x + y}, \\
X &= (x + a^2)(x + b^2) - 2mx, \quad Y = -(a^2 - y)(b^2 - y), \\
U &= yX - xY, \quad Z = ab(X + Y), \\
G &= (x + y)(x + y - 2m). \tag{2.38}
\end{aligned}$$

The results of the three $U(1)$ gauge fields and the scalars are found to be the same for all other known solutions to (2.37), and are given by

$$\begin{aligned}
\mathcal{A}_i &= \frac{2m}{h_i} \left\{ c_i s_i dt + s_i c_j c_k \left[abd\chi + (y - a^2 - b^2)d\sigma \right] \right. \\
&\quad \left. + c_i s_j s_k (abd\sigma - yd\chi) \right\}, \quad i \neq j \neq k, \\
X_i &= \frac{h_1^{1/3} h_2^{1/3} h_3^{1/3}}{h_i}, \quad h_i = x + y + 2m s_i^2, \tag{2.39}
\end{aligned}$$

where $s_i = \sinh \delta_i$, $c_i = \cosh \delta_i$ and $i, j, k = 1, 2, 3$. The Cvetič-Youm solution was found from the neutral Myers-Perry solution by using solution generating techniques [96]. Such a process is possible only for ungauged supergravity theories.

When the cosmological constant is nonzero, there is no solution generating technique can be used and so it is much more difficult to get new solutions. The usual procedure is a lot of trial and brutal force work, combined with some educated guess of the possible right ansatz for the metric and matter fields, based on the form of known solutions. The non-rotating case is always is the easiest and the solution was

found in 1998 [98],

$$\begin{aligned}
ds^2 &= (H_1 H_2 H_3)^{1/3} \left(\frac{dr^2}{f} + r^2 d\Omega_3^2 \right) - \frac{f dt^2}{(H_1 H_2 H_3)^{2/3}}, \\
\mathcal{A}_i &= \frac{2m c_i s_i}{r^2 H_i} dt, \quad H_i = 1 + 2m s_i^2 / r^2, \quad i = 1, 2, 3, \\
f &= 1 - \frac{2m}{r^2} + g^2 r^2 H_1 H_2 H_3.
\end{aligned} \tag{2.40}$$

This solutions contains the cosmological constant, the mass, and three charges characterized by g , m , δ_1 , δ_2 and δ_3 . The rotation was not added until 2004 [99], but in which case the two rotations are set equal, so are the three charges. The case with the two rotations set equal but with three charges arbitrary was found a month later [100]. The solution is given by

$$\begin{aligned}
ds^2 &= (h_1 h_2 h_3)^{1/3} \left[\frac{dr^2}{f} + d\theta^2 + \cos^2 \theta d\phi^2 + \sin^2 \theta d\psi^2 - \frac{1 + g^2(r^2 + 2m\tilde{s}^2)}{r^2 + 2m\tilde{s}^2} dt^2 \right] \\
&\quad + \frac{2m(r^2 + 2m\tilde{s}^2)\ell^2}{(h_1 h_2 h_3)^{2/3}} \left[\mathcal{A} - \frac{r^2 c_1 c_2 c_3 - (r^2 - 2m)s_1 s_2 s_3}{(r^2 + 2m\tilde{s}^2)\ell} dt \right]^2, \\
\mathcal{A}_i &= \frac{2m}{h_i} \left[c_i s_i dt + \ell(c_i s_j s_k - s_i c_j c_k) \mathcal{A} \right], \quad \mathcal{A} = \cos^2 \theta d\phi + \sin^2 \theta d\psi, \\
f &= r^2 - \frac{2m[r^2 - \ell^2 - g^2 \ell^2 (r^2 + 2m\tilde{s}^2)]}{r^2} + \frac{g^2 h_1 h_2 h_3}{r^2}, \\
\tilde{s}^2 &= 2s_1 s_2 s_3 (c_1 c_2 c_3 - s_1 s_2 s_3) - s_1^2 s_2^2 - s_1^2 s_3^2 - s_2^2 s_3^2, \\
X_i &= \frac{h_1^{1/3} h_2^{1/3} h_3^{1/3}}{h_i}, \quad h_i = r^2 + 2m s_i^2.
\end{aligned} \tag{2.41}$$

This solutions contains 6 parameters in total: the cosmological constant, the mass, the three charges and two equal rotations (characterized by the parameter ℓ). Although this solution appears to be only one parameter away from most general solution, adding the last parameter has proved to be a very difficult task. In fact, the most general solution is still unknown.

When one lifts the degeneracy between the rotations, solutions are found only for cases with the charge parameters constrained. In this respect, the first example was

given in [101], where the solution has two arbitrary rotations, while the three nonzero charges are controlled by a single parameter. The same paper also presents another solution with only one non-zero rotation and one non-zero charge. Soon the solution with two independent rotations and three equal charges was found in [102]. The case with two independent rotations and one non-zero charge was found sometime later in [97]. More importantly, it is shown in [97] that all the solutions with two different rotations can be cast in a form similar to (2.38), with the matter fields given by (2.39). What's more, all the gauged solutions in [97] share the common feature that two of their charges are equal. This feature has made it possible for us to construct a more general solution with two independent rotations and two independent charge parameters [21], while all the gauged solutions in [97] have only one charge parameter. Our solution covers all the known examples with the two rotations different. We will discuss this solution in detail in section E.

Apart from those in five dimensions, solutions to supergravity theories in other dimensions have also been found. One can reduce the eleven-dimensional supergravity on an S^7 or S^4 to get supergravities in four and seven dimensions. One can also reduce the ten-dimensional massive Type IIA supergravity on S^4 to get a supergravity theory in six dimensions. Solutions to these theories can be found in [103, 104, 105, 106].

D. Plebanski-Demianski Type Solutions in $d = 5$

In this section, we shall discuss a new Plebanski-Demianski type solution in five dimensions. The solution contains a non-trivial NUT-like parameter. Interesting limits of the solution include the five-dimensional Myers-Perry solution [71], the original Emparan-Reall black ring solution [89], and also solutions with different lens space topologies at the spatial infinity and at the horizon (these solutions demonstrate the

non-uniqueness of black hole solutions in higher dimensions with fixed asymptotic structures). We will also discuss charged version of the solution in the later part of the section.

This work was done with H. Lü and C. N. Pope [22, 23].

1. The General Neutral Solution

Our starting point is an equivalent form of the the five dimensional Kerr-AdS metric given in (2.31),

$$ds_5^2 = \frac{x-y}{4X}dx^2 + \frac{y-x}{4Y}dy^2 + \frac{X(d\phi + yd\psi)^2}{x(x-y)} + \frac{Y(d\phi + xd\psi)^2}{y(y-x)} + \frac{a_0}{xy} \left(d\phi + (x+y)d\psi + xydt \right)^2, \quad (2.42)$$

where

$$X = a_0 + a_1 x + a_2 x^2 + g^2 x^3, \quad Y = a_0 + b_1 y + a_2 y^2 + g^2 y^3. \quad (2.43)$$

The constants a_0 , a_1 , a_2 and b_1 are related to the mass, two angular momenta and the NUT parameter [107, 88]. The NUT parameter can be eliminated by using a coordinate scaling symmetry of the metric [107].

Our strategy is to start with an ansatz similar to the four-dimensional Plebanski-Demianski metric (2.22),

$$ds^2 = f_1(xy) \left[\frac{x-y}{4X}dx^2 + \frac{y-x}{4Y}dy^2 + \frac{X(d\phi + yd\psi)^2}{x(x-y)} + \frac{Y(d\phi + xd\psi)^2}{y(y-x)} \right] + f_2(xy) \frac{a_0}{xy} \left[d\phi + (x+y)d\psi + xydt \right]^2. \quad (2.44)$$

We find a solution in the Ricci flat case,

$$ds_5^2 = \frac{1}{(1-xy)^2} \left[\frac{x-y}{4X}dx^2 + \frac{y-x}{4Y}dy^2 + \frac{X(d\phi + yd\psi)^2}{x(x-y)} + \frac{Y(d\phi + xd\psi)^2}{y(y-x)} \right]$$

$$+\frac{a_0}{xy} \left[d\phi + (x+y)d\psi + xydt \right]^2, \quad (2.45)$$

where

$$\begin{aligned} X &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_0 x^4, \\ Y &= a_0 + a_3 y + a_2 y^2 + a_1 y^3 + a_0 y^4. \end{aligned} \quad (2.46)$$

Sometimes it is helpful to make a change of coordinates,

$$x \rightarrow 1/x, \quad t \rightarrow it, \quad \phi \rightarrow i\phi, \quad \psi \rightarrow i\psi, \quad (2.47)$$

and write the metric as

$$\begin{aligned} ds_5^2 &= \frac{1}{(x-y)^2} \left[\frac{x(1-xy)dx^2}{4G(x)} - \frac{x(1-xy)dy^2}{4G(y)} - \frac{G(x)(d\phi + yd\psi)^2}{(1-xy)} \right. \\ &\quad \left. + \frac{xG(y)(d\psi + xd\phi)^2}{y(1-xy)} \right] - \frac{a_0 y}{x} \left[dt + \frac{x}{y} d\phi + (x+y^{-1})d\psi \right]^2, \end{aligned} \quad (2.48)$$

where

$$G(\xi) = a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3 + a_0 \xi^4. \quad (2.49)$$

Since a Ricci-flat metric remains Ricci-flat when scaled by any constant factor, one can absorb one of the four parameters in (2.49) into an overall dimensionful scale. This then implies that the local metric has three non-trivial parameters in total, which are continuous and dimensionless.

To discuss the global structure of the solution (2.48), let's rewrite (2.49) as

$$G(\xi) = \mu^2 (\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)(\xi - \xi_4), \quad (2.50)$$

with $a_0 = \mu^2$ and $\xi_1 \xi_2 \xi_3 \xi_4 = 1$. The constant μ has dimensions $(\text{length})^{-1}$, whilst ξ_1 , ξ_2 and ξ_3 are the three non-trivial dimensionless parameters. To get a black hole

solution, we let

$$\xi_1 < -1 < \xi_2 < 0 < \xi_3 \leq \xi_4, \quad \xi_1 \xi_2 \leq 1, \quad (2.51)$$

and

$$\xi_1 \leq x \leq \xi_2, \quad \xi_2 \leq y \leq +\infty. \quad (2.52)$$

The asymptotic region is approached at $x = y = \xi_2$, and the outer and inner horizons are at $y = \xi_3$ and $y = \xi_4$ respectively. The surface of the ergosphere is at $y = 0$. The curvature has power-law singularities at $xy = 1$ and at $y = \infty$. The former does not lie in the spacetime manifold, and the latter lies behind the horizons. For later convenience, we shall use $\eta_1 = -\xi_1$ and $\eta_2 = -\xi_2$ instead of ξ_1 and ξ_2 .

We now analyze the conditions needed to prevent any conical singularities outside the horizon. To do this, let's redefine the time coordinate,

$$t \rightarrow t/\mu + (\eta_2 + \eta_2^{-1})\psi - \phi, \quad (2.53)$$

and then introduce new azimuthal angles,

$$\begin{aligned} \phi_1 &= \frac{\mu^2(\eta_1 - \eta_2)(\eta_2 + \xi_3)(1 + \eta_1\eta_2^2\xi_3)}{\eta_1\eta_2^{3/2}\xi_3(1 - \eta_2^2)}(\phi - \eta_2\psi), \\ \phi_2 &= \frac{\mu^2(\eta_1 - \eta_2)(\eta_2 + \xi_3)(1 + \eta_1\eta_2^2\xi_3)}{\eta_1\eta_2^{3/2}\xi_3(1 - \eta_2^2)}(\psi - \eta_2\phi). \end{aligned} \quad (2.54)$$

The metric degenerates at $x = \xi_1 = -\eta_1$, $x = \xi_2 = -\eta_2$ and $y = \xi_2 = -\eta_2$. The three corresponding degenerating Killing vectors, normalized to have unit Euclidean surface gravity (which means that they have 2π periods), are given by $\ell_1 = \partial/\partial\phi_1$ at $x = -\eta_2$, $\ell_2 = \partial/\partial\phi_2$ at $y = -\eta_2$, and

$$\begin{aligned} \ell_3 &= -\frac{\eta_1^{3/2}\xi_3(1 - \eta_1\eta_2)}{\mu(\eta_1 + \xi_3)(1 + \eta_1^2\eta_2\xi_3)}\frac{\partial}{\partial t} \\ &+ \frac{\sqrt{\eta_1}(\eta_2 + \xi_3)(1 + \eta_1\eta_2^2\xi_3)}{\sqrt{\eta_2}(\eta_1 + \xi_3)(1 - \eta_2^2)(1 + \eta_1^2\eta_2\xi_3)} \left[(1 - \eta_1\eta_2)\frac{\partial}{\partial\phi_1} + (\eta_1 - \eta_2)\frac{\partial}{\partial\phi_2} \right] \end{aligned}$$

at $x = -\eta_1$. Since each ℓ_i independently generates a 2π translation around its degeneration surface, it follows in particular that the time coordinate t must be periodic, which is a property of any Taub-NUT solution. Note that the special case $\eta_1\eta_2 = 1$ describes the Myers-Perry rotating black hole, which does not have the time periodicity. This is consistent with the fact that the $\partial/\partial t$ term disappears from ℓ_3 in this particular case.

To have a look at the asymptotic region, located at $x = y = \xi_2 = -\eta_2$, we use the radial coordinates,

$$\begin{aligned} \frac{\sqrt{\xi_2 - x}}{y - x} &= A r \cos \theta, & \frac{\sqrt{y - \xi_2}}{y - x} &= A r \sin \theta, \\ A^2 &= \frac{\mu^2(\eta_1 - \eta_2)(\eta_2 + \xi_3)(1 + \eta_1\eta_2^2\xi_3)}{\eta_1\eta_2^2\xi_3(1 - \eta_2^2)}. \end{aligned} \quad (2.55)$$

Taking the limit $r \rightarrow \infty$, we see that (2.48) approaches

$$ds = -dt^2 + dr^2 + r^2(d\theta^2 + \cos^2\theta d\phi_1^2 + \sin^2\theta d\phi_2^2). \quad (2.56)$$

For thermodynamical quantities, the mass and the angular momenta can be evaluated using the Komar integral,

$$\begin{aligned} M &= \frac{3\pi\eta_1\eta_2\xi_3(1 - \eta_2^2)}{8\mu^2(\eta_1 - \eta_2)(\eta_2 + \xi_3)(1 + \eta_1\eta_2^2\xi_3)}, \\ J_{\phi_2} &= \frac{\pi\eta_1^2\eta_2^{3/2}\xi_3^2(1 - \eta_2^2)^2}{4\mu^3(\eta_1 - \eta_2)^2(\eta_2 + \xi_3)^2(1 + \eta_1\eta_2^2\xi_3)^2}, & J_{\phi_1} &= \eta_2 J_{\phi_2}. \end{aligned} \quad (2.57)$$

The outer horizon is at $y = \xi_3$. The asymptotically timelike Killing vector that degenerates there is given by

$$\ell_0 = \frac{\partial}{\partial t} - \frac{\mu(\eta_1 - \eta_2)(1 + \eta_1\eta_2^2\xi_3)}{\eta_1\sqrt{\eta_2}(1 + \eta_2\xi_3)(1 - \eta_2^2)} \left[(1 + \eta_2\xi_3) \frac{\partial}{\partial \phi_2} - (\eta_2 + \xi_3) \frac{\partial}{\partial \phi_1} \right]. \quad (2.58)$$

From this, the temperature, the entropy and the angular velocities are given by

$$\begin{aligned}
T &= \frac{\mu(\eta_1 + \xi_3)(1 - \eta_1\eta_2\xi_3^2)}{2\pi\eta_1\sqrt{\xi_3}(1 + \eta_2\xi_3)}, \\
S &= \frac{\pi^2(\eta_1\eta_2)^2\xi_3^{3/2}(1 + \eta_2\xi_3)(1 - \eta_2^2)}{2\mu^3(\eta_1 + \xi_3)(\eta_1 - \eta_2)(\eta_2 + \xi_3)^2(1 + \eta_1\eta_2^2\xi_3)^2}, \\
\Omega_{\phi_1} &= \frac{\mu(\eta_1 - \eta_2)(\eta_2 + \xi_3)(1 + \eta_1\eta_2^2\xi_3)}{\eta_1\sqrt{\eta_2}(1 + \eta_2\xi_3)(1 - \eta_2^2)}, \\
\Omega_{\phi_2} &= -\frac{\mu(\eta_1 - \eta_2)(1 + \eta_1\eta_2^2\xi_3)}{\eta_2\sqrt{\eta_2}(1 - \eta_2^2)}. \tag{2.59}
\end{aligned}$$

The first law of thermodynamics is not satisfied for this solution. This can be related to the fact that the analysis of the thermodynamics of Taub-NUT solutions is notoriously unsettled. However, the first law is satisfied if $\eta_1\eta_2 = 1$,

$$dM = TdS + \Omega_{\phi_1}dJ_{\phi_1} + \Omega_{\phi_2}dJ_{\phi_2}, \quad M = \frac{3}{2}(TS + \Omega_{\phi_1}J_{\phi_1} + \Omega_{\phi_2}J_{\phi_2}). \tag{2.60}$$

In fact when $\eta_1\eta_2 = 1$, the metric is nothing but the Myers-Perry rotating black hole, in an unusual coordinate system. We will see later that the Myers-Perry solution also arises by taking a certain limit of the general solution (2.45). The coordinate transformation linking these two results is quite complicated, and we have verified their equivalence by studying the relationship between the mass, entropy and angular momenta.

Finally we remark that although our solution is similar to four-dimensional Taub-NUT spacetimes in that the time coordinate is periodic, there are also significant differences. In four dimensions, the time direction is fibered over the two sphere and this is what's responsible for imposing the periodicity of the time coordinate. For our solution, the metric approaches Minkowski spacetime locally at infinity and there is no fibration in the time direction. Our solution is also very different in structure from the topological soliton “time machines” obtained in [108], where there are no

horizons or singularities in the spacetime. By contrast, our solution describes black objects containing horizons, and with singularities inside the horizons. Note, further, that the outer horizon $y = \xi_3$ of our solution is separated from the velocity of light surface surrounding the time machine at $y = \xi_2 = -\eta_2$.

2. Static Black Holes with Lens Space Topology

There are various interesting limits one can take for the general solution (2.45) or (2.48). For example, starting from (2.45), one can make the following scaling,

$$\begin{aligned} x &\rightarrow \epsilon^2 x, & y &\rightarrow \epsilon^2 y, & \phi &\rightarrow \epsilon^{-1} \phi, & \psi &\rightarrow \epsilon^{-3} \psi, & t &\rightarrow \epsilon^{-5} t, \\ a_0 &\rightarrow \epsilon^6 a_0, & a_1 &\rightarrow \epsilon^4 a_1, & a_2 &\rightarrow \epsilon^2 a_2, & a_3 &\rightarrow \epsilon^4 a_3. \end{aligned} \quad (2.61)$$

Upon sending $\epsilon \rightarrow 0$, the metric reduces to (2.42), with

$$X = a_0 + a_1 x + a_2 x^2, \quad Y = a_0 + a_3 y + a_2 y^2. \quad (2.62)$$

This is just the five-dimensional Myers-Perry black hole, in the form given in [107, 88]. It has two non-trivial continuous (dimensionless) parameters. (One of the four parameters in (2.62) can be absorbed by means of a coordinate transformation, and a second by making an overall constant scaling of the metric.)

As a second example, one can start with (2.48) and let

$$\begin{aligned} x &\rightarrow \epsilon^2 x, & y &\rightarrow \epsilon^2 y, & \phi &\rightarrow \epsilon \phi, & \psi &\rightarrow \epsilon \psi, & t &\rightarrow \epsilon^{-1} t, \\ a_0 &\rightarrow \epsilon^2 a_0, & a_1 &\rightarrow a_1, & a_2 &\rightarrow \epsilon^{-2} a_2, & a_3 &\rightarrow \epsilon^{-4} a_3. \end{aligned} \quad (2.63)$$

Upon sending $\epsilon \rightarrow 0$, this leads to the metric

$$ds_5^2 = \frac{1}{(x-y)^2} \left[\frac{x dx^2}{4G(x)} - \frac{x dy^2}{4G(y)} - G(x) d\phi^2 + \frac{x G(y) d\psi^2}{y} \right]$$

$$-\frac{a_0 y}{x} \left(dt + y^{-1} d\psi \right)^2, \quad (2.64)$$

$$G(\xi) = a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3. \quad (2.65)$$

One of the four parameters in (2.65) can be absorbed by means of a coordinate transformation, and a second can be absorbed by making an overall constant scaling of the metric. So in this case the solution has two non-trivial parameters, which are continuous and dimensionless. It can be shown [22] that the metric (2.64) contains the black ring found in [89]. So this limit is a five-dimensional analogue of the limit in which the Plebanski-Demianski metric gives rise to the C-metric in four dimensions (see, for example, [85]). In fact, the local black ring solution was obtained from Wick rotation of the Kaluza-Klein lifting [109] of a dilatonic generalization of the four-dimensional C-metric [110].

Our third example leads to a wide class of static black holes. For this, we start with (2.48) and make the following scaling

$$a_0 \rightarrow \epsilon^2 a_0, \quad a_0^{1/2} t \rightarrow \epsilon^{-1} t, \quad \phi \rightarrow i\phi, \quad \psi \rightarrow i\psi. \quad (2.66)$$

Upon sending ϵ to zero, we obtain the metric

$$ds_5^2 = \frac{1}{(x-y)^2} \left[\frac{(1-xy)dx^2}{4G(x)} - \frac{x(1-xy)dy^2}{4yG(y)} + \frac{xG(x)(d\phi + yd\psi)^2}{(1-xy)} - \frac{xG(y)(d\psi + xd\phi)^2}{(1-xy)} \right] - \frac{y}{x} dt^2, \quad (2.67)$$

where

$$G(\xi) = a_1 + a_2 \xi + a_2 \xi^2. \quad (2.68)$$

Note that the Wick rotations of ϕ and ψ in (2.67) are performed just for later convenience; the same effect could be achieved by sending $x \rightarrow -x$, $y \rightarrow -y$ and $\psi \rightarrow -\psi$. One of the three parameters in (2.68) can be absorbed by an overall constant scaling of

the metric, and so the metric (2.67) has two non-trivial continuous and dimensionless parameters.

To discuss the global structure of the metric, we rewrite (2.68) as

$$G(\xi) = -\mu^2(\xi - \xi_1)(\xi - \xi_2). \quad (2.69)$$

We require that

$$0 < \xi_1 \leq \xi_2, \quad \xi_1 \xi_2 \leq 1, \quad (2.70)$$

and

$$\xi_1 \leq x \leq \xi_2, \quad -\infty \leq y \leq \xi_1. \quad (2.71)$$

The asymptotic region at infinity occurs at $x = \xi_1 = y$, and the horizon is located at $y = 0$. There is a power-law singularity at $y = \infty$, which is hidden by the horizon when the ξ_i parameters are chosen as described above. There would also be a power-law singularities at $x = 0$ and $xy = 1$, but these do not lie within the spacetime manifold, for the choice of coordinate ranges and parameters we are making. The metric contains no closed time-like circles outside the horizon. There are two special cases. One is when $\xi_1 \xi_2 = 1$, for which the solution reduces to standard five-dimensional Schwarzschild-Tangherlini black hole. The other is when $\xi_1 = \xi_2$, for which the solution describes the Kaluza-Klein monopole. We will focus on the case when

$$0 < \xi_1 < \xi_2, \quad \xi_1 \xi_2 < 1. \quad (2.72)$$

Now the metric (2.67) degenerates at three locations $x = \xi_1$, $x = \xi_2$ and $y = \xi_1$. The three corresponding Killing vectors, normalized to have unit a Euclidean surface gravity (which means that each has a 2π period), are given by

$$x = \xi_1 : \quad \ell_1 = \frac{\partial}{\partial \phi_1},$$

$$\begin{aligned}
x = \xi_2 : \quad \ell_2 &= \alpha \frac{\partial}{\partial \phi_1} + \beta \frac{\partial}{\partial \phi_2}, \\
y = \xi_1 : \quad \ell_3 &= \frac{\partial}{\partial \phi_2},
\end{aligned} \tag{2.73}$$

where

$$\alpha = \frac{(1 - \xi_1 \xi_2) \sqrt{\xi_1}}{(1 - \xi_1^2) \sqrt{\xi_2}}, \quad \beta = -\frac{(\xi_2 - \xi_1) \sqrt{\xi_1}}{(1 - \xi_1^2) \sqrt{\xi_2}}, \tag{2.74}$$

and we have defined two new azimuthal coordinates,

$$\phi_1 = \frac{\mu^2 \sqrt{\xi_1} (\xi_2 - \xi_1) (\phi + \xi_1 \psi)}{1 - \xi_1^2}, \quad \phi_2 = \frac{\mu^2 \sqrt{\xi_1} (\xi_2 - \xi_1) (\psi + \xi_1 \phi)}{1 - \xi_1^2}. \tag{2.75}$$

The three Killing vectors ℓ_1, ℓ_2 and ℓ_3 span a two-dimensional vector space, and so there is a linear relation between them. A necessary condition for avoiding conical singularities is that the coefficients in this linear relation must be rationally related, since otherwise it would be possible, by taking integer combinations of 2π rotations around the circles, to generate a translation that implied an identification of arbitrarily close points on the manifold. By an overall scaling in the linear relation, one gets that the coefficients must be coprime integers [111, 23],

$$p\ell_1 + m\ell_2 + n\ell_3 = 0. \tag{2.76}$$

Furthermore, note that ℓ_2 and ℓ_3 can be simultaneously degenerate when $x = \xi_2$ and $y = \xi_1$, which implies that any linear combination of ℓ_2 and ℓ_3 is also a degenerate Killing vector at this surface. For the coprime integer pair (m, n) , the minimum period generated by $m\ell_2 + n\ell_3$ is 2π . It follows that in order to avoid a conical singularity, we must have $p = \pm 1$. Without loss of generality, let $p = -1$, and hence

$$\ell_1 = m\ell_2 + n\ell_3. \tag{2.77}$$

It follows that

$$\frac{(1 - \xi_1^2)\sqrt{\xi_2}}{(1 - \xi_1\xi_2)\sqrt{\xi_1}} = m, \quad \frac{\xi_2 - \xi_1}{1 - \xi_1\xi_2} = n. \quad (2.78)$$

Thus the solution space is parameterized by a pair of coprime integers (m, n) . For the parameter range specified in (2.72), the integers (m, n) must obey the inequalities

$$m \geq n + 2 \geq 3. \quad (2.79)$$

The case $n = 1$ occurs when $\xi_2 = 1$.

A careful discussion [22] can show that, on the horizon, the (only) two degenerate Killing vectors are ℓ_1 and ℓ_2 , giving rise to a geometry of non-homogeneously distorted lens space $L(n; m)$; while in the asymptotic region, the (only) two degenerate Killing vectors are (ℓ_1, ℓ_3) , and the large- r spatial sections have the geometry of homogeneous lens spaces $L(m; n)$. When the global structure is clear, it is then a straightforward matter to calculate the entropy, temperature and mass of the black holes,

$$\begin{aligned} S &= \frac{\pi^2(1 - \xi_1\xi_2)}{2\mu^3\xi_1\xi_2(\xi_2 - \xi_1)}, & T &= \frac{\mu\sqrt{\xi_1\xi_2}}{2\pi}, \\ M &= \frac{3\pi(1 - \xi_1\xi_2)}{8\mu^2(\xi_2 - \xi_1)\sqrt{\xi_1\xi_2}}. \end{aligned} \quad (2.80)$$

It is easily verified that the black holes satisfy the first law of thermodynamics, $dM = TdS$. Furthermore, one also has $M = \frac{3}{2}TS$, as in the case of five dimensional Schwarzschild black hole. It should be noted that in this calculation, involving an integration over lens spaces at the spatial infinity or at the horizon, one must take into account the periodicity conditions implied by the lens-space identifications. The general rule is that when a given 3-sphere metric is factored to give the lens space $L(p; q)$, the 3-volume is reduced by a factor of $1/p$.

It should be emphasized that these results do not contradict results on the uniqueness of higher-dimensional static asymptotically-flat black holes in [112]. Since the

spatial sections at large distance in our new solutions have the topology of the $L(m; n)$ lens space, which is the quotient of S^3 by a certain discrete subgroup $\Gamma(m; n)$ of $SO(4)$, it follows that although the curvature tends to zero at infinity the spacetime is not asymptotic to Minkowski spacetime, but, rather, to the quotient Minkowski/ $\Gamma(m; n)$. Thus the conditions assumed in [112], under which uniqueness could be proved, are not satisfied.

One can also, of course, consider a different and considerably simpler static black hole with the same asymptotic geometry Minkowski/ $\Gamma(m; n)$. As was noted in [112], the round S^n in any $D = n + 2$ dimensional Schwarzschild-Tangherlini solution can be replaced by an arbitrary Einstein space of the same Ricci curvature. Although the five-dimensional example was not discussed explicitly in [112], one can simply replace S^3 in the five-dimensional Schwarzschild-Tangherlini spacetime by the lens space $L(m; n)$. In this case, unlike our new solutions, the horizon will have the same round $L(p; q)$ lens space geometry as the large- r spatial sections. There are only two zero-length Killing vectors in the whole metric. These factored Schwarzschild-Tangherlini solutions are of cohomogeneity 1, in contrast to our new solutions, which have cohomogeneity 2. For each of the new solutions with asymptotic $L(m; n)$ spatial sections that we have obtained in this paper, there is another, inequivalent, black hole with the same asymptotic structure, obtained instead by simply factoring the S^3 in the Schwarzschild-Tangherlini solution by $\Gamma(m; n)$.

One way to compare the different black-hole metrics is to look at the dimensionless quantity obtained by multiplying the entropy by the cube of the temperature,

$$S = \frac{1}{16\pi T^3} \frac{\sqrt{\xi_1 \xi_2}}{n}. \quad (2.81)$$

At fixed temperature, therefore, the entropy is maximized by the Schwarzschild-Tangherlini spacetime, which corresponds to $m = n = 1$ and $\xi_1 \xi_2 = 1$. It is interesting

to note that the “factored Schwarzschild-Tangherlini” solution, in which S^3 surfaces are quotiented to give $L(m; n) = S^3/\Gamma(m; n)$, will have a smaller entropy than our new “slumped” black hole with $L(n; m)$ horizon topology. This follows from the fact that the former will have entropy $S = 1/(16m\pi T^3)$, whereas the slumped solution has entropy given by (2.81), which is larger by the factor

$$1 + \frac{\xi_1 (1 - \xi_1 \xi_2)}{\xi_2 - \xi_1} \geq 1. \quad (2.82)$$

One further remark concerns the limit $\xi_1 \xi_2 \rightarrow 1$, which gives the usual Schwarzschild-Tangherlini metric. It might appear that the mass formula (2.80) is incompatible with this limit, since it vanishes when $\xi_1 \xi_2 = 1$. To resolve this apparent paradox, we note that when $\xi_1 \xi_2 = 1$, it follows that $\ell_2 = -\partial/\partial\phi_2$. Thus (2.76) can be simply solved by letting $p = 0$ and $m = n = 1$. Then the condition (2.77) no longer holds, and ϕ_1 and ϕ_2 both have independent 2π periods. The solution indeed describes the standard Schwarzschild black hole. However, within our general class of black-hole solutions, taking the limit $\xi_1 \xi_2 \rightarrow 1$ assumes that the condition (2.77) is still imposed. This corresponds to sending m and n to infinity, while keeping $m/n \rightarrow 1$. The resulting metric then describes a Schwarzschild-Tangherlini black hole in which the round S^3 is replaced by $S^3/\Gamma(\infty; \infty)$. This has zero volume, and so the mass would vanish too.

3. The Charged Solutions

Since the solutions in the previous subsections are Ricci flat, solution generating techniques in ungauged supergravity theories can be used to add charges to these solutions.

a. Electromagnetically Charged Solutions

To do this, we start with the electrically-charged rotating black hole solution of five-dimensional minimal supergravity,

$$\begin{aligned}
ds^2 &= (x-y) \left(\frac{dx^2}{4X} - \frac{dy^2}{4Y} \right) - \frac{X(dt + yd\phi)^2}{x(x-y)} + \frac{Y(dt + xd\phi)^2}{y(x-y)}, \\
&\quad - \frac{1}{xy} \left[\left(\mu - \frac{qy}{x-y} \right) dt + (x+y) \left(\mu - \frac{qy^2}{x^2-y^2} \right) d\phi + xy d\chi \right]^2, \\
A &= \frac{\sqrt{3}q}{(x-y)} (dt + yd\phi),
\end{aligned} \tag{2.83}$$

where

$$X = (\mu + q)^2 + a_3 x + a_2 x^2, \quad Y = \mu^2 + a_1 y + a_2 y^2. \tag{2.84}$$

This is a special case of the Cvetič-Youm solution given in (2.38), with the three $U(1)$ charges set equal. If the charge parameter is set to zero, $q = 0$, this solution reduces to the five-dimensional Myers-Perry black hole (2.28).

Now one can use an ansatz similar to (2.44) and try to solve for the solution.

But doing that, it will be convenient to make the redefinition,

$$t \longrightarrow \phi, \quad \phi \longrightarrow \psi, \quad \chi \longrightarrow t, \quad x \longrightarrow \frac{1}{x}. \tag{2.85}$$

After this, the new solution is found to be

$$\begin{aligned}
ds^2 &= \frac{1}{(x-y)^2} \left[x(1-xy) \left(\frac{dx^2}{4G(x)} - \frac{dy^2}{4G(y)} \right) - \frac{G(x)(d\phi + yd\psi)^2}{1-xy} \right. \\
&\quad \left. + \frac{xG(y)(d\psi + xd\phi)^2}{y(1-xy)} \right] \\
&\quad - \frac{y}{x} \left[dt + \frac{x}{y} \left(\mu - \frac{qxy}{1-xy} \right) d\phi + (x+y^{-1}) \left(\mu - \frac{qx^2y^2}{1-x^2y^2} \right) d\psi \right]^2, \\
A &= \frac{\sqrt{3}qx}{1-xy} (d\phi + yd\psi),
\end{aligned} \tag{2.86}$$

where

$$G(\xi) = \mu^2 + a_1\xi + a_2\xi^2 + a_3\xi^3 + (\mu + q)^2\xi^4. \quad (2.87)$$

This solution goes back to (2.83) by using the following scaling

$$\begin{aligned} x &\longrightarrow \frac{1}{\epsilon^2 x}, & y &\longrightarrow \epsilon^2 y, & \phi &\longrightarrow \epsilon^{-1} t, & t &\longrightarrow \epsilon^{-2} \chi, & \psi &\longrightarrow \phi, \\ \mu &\longrightarrow \epsilon^3 \mu, & q &\longrightarrow \epsilon^3 q, & a_1 &\longrightarrow \epsilon^4 a_1, & a_2 &\longrightarrow \epsilon^2 a_2, & a_3 &\longrightarrow \epsilon^4 a_3, \end{aligned} \quad (2.88)$$

and then let $\epsilon \rightarrow 0$.

The solution (2.86) carries a purely magnetic dipole-like charge². One can add to it an electric charge by reducing the solution on the time direction, performing an $O(1, 1)$ U-duality transformation, and then lifting it back to five dimensions. The detail of this process can be found in [23]. The new solution is then

$$\begin{aligned} ds_5^2 &= H ds_4^2 - \frac{y}{xH^2} (dt + w)^2, & H &= c^2 - \frac{s^2 y}{x}, \\ w &= -\frac{q(s + cx)^3 (d\phi + y d\psi)}{x(1 - xy)} + \frac{(\mu + q)s^3 [(1 + xy)d\phi + y d\psi]}{x} \\ &\quad + \frac{c^3 \mu [(1 + xy)d\psi + x d\phi]}{y}, \\ A &= \frac{\sqrt{3}}{xH} \left\{ cs(x - y)dt + \frac{q(s + cx)^2 (c + sy)(d\phi + y d\psi)}{(1 - xy)} \right. \\ &\quad \left. - (\mu + q)cs^2 [(1 + xy)d\phi + y d\psi] - \mu c^2 s [(1 + xy)d\psi + x d\phi] \right\}, \end{aligned} \quad (2.89)$$

where

$$ds_4^2 = \frac{1}{(x - y)^2} \left[x(1 - xy) \left(\frac{dx^2}{4G(x)} - \frac{dy^2}{4G(y)} \right) \right]$$

²As is shown in the appendix B of [23], in a certain limit (2.86) reduces to the black ring [113] with its standard magnetic dipole charge. In our new solutions it is less clear how to interpret the magnetic charge, but we shall continue, for the sake of brevity, to refer to it as a dipole charge since it does acquire this interpretation in the black ring limit.

$$G(\xi) = (\mu + q)^2(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)(\xi - \xi_4), \quad (2.90)$$

$$-\frac{G(x)(d\phi + yd\psi)^2}{1 - xy} + \frac{xG(y)(d\psi + xd\phi)^2}{y(1 - xy)} \Big],$$

with $\xi_1\xi_2\xi_3\xi_4 = \mu^2/(\mu + q)^2$.

To study the global structure of the solution, one requires $\xi_1 < \xi_2 < 0 < \xi_3 < \xi_4$, with $\xi_1\xi_2 < 1$, and

$$\xi_1 \leq x \leq \xi_2, \quad \xi_2 \leq y \leq \xi_3. \quad (2.91)$$

Asymptotic infinity is located at $x = \xi_2 = y$, and the horizon is at $y = \xi_3$, with an ergosphere at $y = 0$. It will be useful to shift the time coordinate,

$$t \rightarrow t - \left[3c^2sq - c^3(\mu + q)(\eta_1 + \eta_2) + \frac{\mu s^3}{\eta_1\eta_2} \right] \psi + \left[3cs^2q - c^3(\mu + q)\eta_1\eta_2 + \frac{\mu s^3(\eta_1 + \eta_2)}{\eta_1\eta_2} \right] \phi, \quad (2.92)$$

where $\eta_1 = -\xi_1$ and $\eta_2 = -\xi_2$. In order to avoid naked closed time-like curves, one must have the constraint

$$q = \frac{\mu(s^3 + c^3\eta_1)(1 - \eta_1\eta_2)(1 - \eta_2^2)}{\eta_1\eta_2 \left[c^3(\eta_1 + \eta_2 - \eta_1\eta_2^2) - s(3c^2 - 3cs\eta_2 + s^2\eta_2^2) \right]}. \quad (2.93)$$

Note that if we turn off q , we can have either $\eta_1\eta_2 = 1$ or $\eta_1 = -s^3/c^3$. The former case leads to a charged rotating black hole solution, whilst the latter case (which requires $s < 0$ since η_1 is positive), leads to a black hole solution with purely electric charge. Note also that if we instead turn off the electric charge (by taking $\sinh \delta = 0$), then (2.93) gives the condition for avoiding CTCs in the solutions (2.86) that carry no electric charge.

The analysis of the global properties of the solution rests upon investigating the behavior at the singular points of the metric, which occur when x or y approach their endpoints. In the present case, these degenerations occur at $x = \xi_1$, $x = \xi_2$ and

$y = \xi_2$. The associated Killing vectors will be normalized so that they have a unit Euclidean surface gravity. They are given by

$$\begin{aligned} x = \xi_1 = -\eta_1 : \quad \ell_1 &= \alpha \left[(1 - \eta_1 \eta_2) \frac{\partial}{\partial \phi_1} - (\eta_1 - \eta_2) \frac{\partial}{\partial \phi_2} \right], \\ y = \xi_2 = -\eta_2 : \quad \ell_2 &= \frac{\partial}{\partial \phi_2}, \\ x = \xi_2 = -\eta_2 : \quad \ell_3 &= \frac{\partial}{\partial \phi_1}, \end{aligned} \tag{2.94}$$

where

$$\begin{aligned} \psi &= \nu(\eta_2 \phi_1 - \phi_2), \quad \phi = \nu(\phi_1 - \eta_2 \phi_2), \\ \nu &= \frac{\sqrt{\eta_2}}{(\mu + q)^2(\eta_2 + \xi_3)(\eta_2 + \xi_4)(\eta_1 - \eta_2)}, \\ \alpha &= \frac{\sqrt{\eta_1}(\eta_2 + \xi_3)(\eta_2 + \xi_4)}{\sqrt{\eta_2}(\eta_1 + \xi_3)(\eta_1 + \xi_4)(1 - \eta_2^2)}. \end{aligned} \tag{2.95}$$

Because of the chosen normalization, if each Killing vector ℓ_i is written in terms of a coordinate ψ_i as $\ell_i = \partial/\partial\psi_i$, then advancing ψ_i by an interval 2π will generate one complete rotation around the origin in the plane of the degeneration. Similar to (2.76), and notice that ℓ_1 and ℓ_2 can degenerate simultaneously, we get

$$\ell_3 = m \ell_1 + n \ell_2. \tag{2.96}$$

where m and n are coprime integers,

$$m = \frac{1}{\alpha(1 - \eta_1 \eta_2)}, \quad n = \frac{\eta_1 - \eta_2}{1 - \eta_1 \eta_2}, \quad 1 \leq n \leq m - 1. \tag{2.97}$$

Now one can write

$$\ell_1 = \frac{1}{m} \frac{\partial}{\partial \phi_1} - \frac{n}{m} \frac{\partial}{\partial \phi_2}. \tag{2.98}$$

A careful discussion [23] can then show that at the spatial infinity, the solution describes lens spaces $L(m; n) = S^3/\Gamma(m; n)$, while the horizon is topologically the lens

spaces $L(n; m)$. Given the global structure of the solution, all the thermodynamical quantities can also be calculated [23].

b. The Black Ring Limit

Interesting limits can be taken for the general solution in (2.89). For example, one can take

$$\begin{aligned} x &\rightarrow \epsilon^2 x, & y &\rightarrow \epsilon^2 y, & \psi &\rightarrow \epsilon \psi, & \phi &\rightarrow \epsilon \phi, \\ q &\rightarrow \epsilon^{-3} q, & \mu &\rightarrow \epsilon \mu, & \xi_i &\rightarrow \xi_i \epsilon^2, & t &\rightarrow t + 3cs^2 q \epsilon^{-1} \phi, \end{aligned} \quad (2.99)$$

and then send $\epsilon \rightarrow 0$. When a divergent pure gauge term in the $U(1)$ potential is discarded, this leads to the solution

$$\begin{aligned} ds^2 &= \frac{H}{(x-y)^2} \left[\frac{xdx^2}{4G(x)} - \frac{xdy^2}{4G(y)} - G(x)d\phi^2 + \frac{xG(y)d\psi^2}{y} \right] - \frac{y}{xH^2}(dt+w)^2, \\ A &= \frac{\sqrt{3}}{xH} \left\{ cs(x-y)dt + \left[qs^3y^2 - c^2s(\mu - 2qxy) \right] d\psi \right. \\ &\quad \left. + \left[c^3qx^2 - cs^2(\mu - 2qxy) \right] d\phi \right\}, \\ w &= \left(\frac{c^3\mu}{y} - 3cs^2qy \right) d\psi + \left(\frac{s^3\mu}{x} - 3c^2sqx \right) d\phi, \quad H = c^2 - \frac{s^2y}{x}, \\ G(\xi) &= q^2(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)(\xi - \xi_4), \end{aligned} \quad (2.100)$$

where the four roots of $G(\xi)$ satisfy $\xi_1\xi_2\xi_3\xi_4 = \mu^2/q^2$. For the global structure, we require that

$$\xi_1 < \xi_2 < 0 < \xi_3 < \xi_4, \quad -\eta^2 \leq x \leq -1, \quad -1 \leq y \leq \infty. \quad (2.101)$$

Since the solution is obtained from a scaling limit, it follows that there is a residual scaling symmetry which enables us to set, without loss of generality, $\xi_2 = -1$. We also define $\eta = \sqrt{-\xi_1} > 1$. There is a region corresponding to asymptotic infinity

located at $x = -1 = y$. Outer and inner horizons are located at $y = \xi_3$ and $y = \xi_4$ respectively. In order for $g_{\psi\psi}$ and $g_{\phi\phi}$ to be non-negative on the degenerate surfaces $x = -\eta^2$, $x = -1$ and $y = -1$, it is necessary to shift the time coordinate t ,

$$t \rightarrow t + c(c^2\mu - 3qs^2)\psi - s(3c^2q - \mu s^2)\phi, \quad (2.102)$$

and also to impose the additional constraint

$$s(\mu s^2 + 3c^2q\eta^2) = 0. \quad (2.103)$$

This leads to a bifurcation of solutions:

$$s = 0, \quad \text{or} \quad q = -\frac{\mu s^2}{3c^2\eta^2}. \quad (2.104)$$

The first case corresponds to turning off the electric charge, and describes a magnetic dipole-charged ring [113]. The second case has a non-vanishing electric charge, and describes a electrically charged magnetic dipole ring [114]. The thermodynamical properties of these solutions are discussed in [23].

c. A Three Charge Solution

The solution in (2.45) can also be viewed as a neutral solution to the STU model (2.35). Since it only solves the ungauged theory, one can use the usual solution generating technique to charge it up. When three independent electric charges are added, the solution is given by

$$\begin{aligned} ds^2 &= (H_1 H_2 H_3)^{1/3} ds_4^2 - \frac{xy(dt + w)^2}{(H_1 H_2 H_3)^{2/3}}, \\ ds_4^2 &= \frac{x - y}{(1 - xy)^2} \left[\frac{dx^2}{4X} - \frac{dy^2}{4Y} - \frac{X(d\phi + yd\psi)^2}{x(x - y)^2} + \frac{Y(d\phi + xd\psi)^2}{y(x - y)^2} \right], \\ w &= \frac{\mu c_1 c_2 c_3 (d\phi + (x + y)d\psi)}{xy} - (\mu + q)s_1 s_2 s_3 (d\phi(x + y) + xy d\psi) \end{aligned}$$

$$\begin{aligned}
& - \frac{q(c_1 - s_1x)(c_2 - s_2x)(c_3 - s_3x)(d\phi + yd\psi)}{x(x - y)} ; \\
X &= (\mu + q)^2 + a_3x + a_2x^2 + a_1x^3 + \mu^2x^4 ; \\
Y &= \mu^2 + a_1y + a_2y^2 + a_3y^3 + (\mu + q)^2y^4 .
\end{aligned} \tag{2.105}$$

The gauge and scalar fields are given by

$$\begin{aligned}
A_i &= \frac{c_i s_i (1 - xy) dt + (\mu + q) c_i s_j s_k (d\phi(x + y) + xy d\psi) - \mu s_i c_j c_k (d\phi + (x + y) d\psi)}{H_i} \\
& - \frac{q(c_i - s_i y)(c_j - s_j x)(c_k - s_k x)(d\phi + yd\psi)}{(x - y) H_i} , \\
H_i &= c_i^2 - s_i^2 xy ; \quad i \neq j \neq k \quad \text{and} \quad i, j, k = 1, 2, 3 .
\end{aligned} \tag{2.106}$$

Here, we have defined $c_i = \cos \delta_i$ and $s_i = \sinh \delta_i$, where δ_i are the three boost parameters giving rise to the three electric charges.

E. A Rotating Non-Extremal Solution to the $d = 5$ Maximal Supergravity

The work described in this section was done with C. N. Pope [21].

From the standpoint of the AdS/CFT correspondence, the most interesting system to look for solutions is the the maximal $SO(6)$ -gauged $\mathcal{N} = 8$ supergravity in $d = 5$. This theory can be obtained by the Pauli reduction of the type IIB superstring on S^5 . In this theory, black holes with Abelian gauge fields can carry 3 independent charges, associated with the three $U(1)$ factors in the Cartan subgroup of $SO(6)$. Equivalently, one can think of such charged black holes as solutions of $\mathcal{N} = 2$ gauged five-dimensional supergravity coupled to two additional vector multiplets. The three charges are carried by the two vectors and the graviphoton of the $\mathcal{N} = 2$ supergravity itself. This system is sometimes called the STU model and the action is given in (2.35). The general black-hole solution should then be characterized by its mass, the

Table 1. All currently known solutions to the STU model.

Gauged	Rotations	Charges	Authors
\times	$a \neq b$	$q_1 \neq q_2 \neq q_3$	Cvetič, Youm [96]
\checkmark	$a = b = 0$	$q_1 \neq q_2 \neq q_3$	Behrndt, Cvetič, Sabara [98]
\checkmark	$a = b \neq 0$	$q_1 = q_2 = q_3 \neq 0$	Cvetič, Lü, Pope [99]
\checkmark	$a = b \neq 0$	$q_1 \neq q_2 \neq q_3$	Cvetič, Lü, Pope [100]
\checkmark	$a \neq b$	$q_1 = q_2, q_3 = q_3(q_1)$	Chong, Cvetič, Lü, Pope [101]
\checkmark	$a \neq b$	$q_1 = q_2 = q_3 \neq 0$	Chong, Cvetič, Lü, Pope [102]
\checkmark	$a \neq b$	$q_1 = q_2 = 0, q_3 \neq 0$	Chong, Cvetič, Lü, Pope [97]
\checkmark	$a \neq b$	$q_1 = q_2 \neq q_3 \neq 0$	Mei, Pope [21]

two independent angular momenta associated with rotations in the two orthogonal spatial 2-planes, and the three independent charge parameters. All currently known solutions to the system are listed in Table 1.

Just one step away from becoming the most general solution, our solution has two independent rotations and but only two independent charge parameters. This corresponds to the situation where two of the three charges in the general solution are set equal, whilst the third can be independently specified. For appropriate specializations of the charge parameters, our our solution recovers all the previous cases with independent rotations [101, 102, 97]. In fact, our solution was obtained by trying to arrange the known solutions [101, 102, 97] into a common form. The result is given by

$$ds^2 = H_1^{2/3} H_3^{1/3} \left\{ (x^2 - y^2) \left(\frac{dx^2}{X} - \frac{dy^2}{Y} \right) - \frac{x^2 X (dt + y^2 d\sigma)^2}{(x^2 - y^2) f H_1^2} \right.$$

$$+ \frac{y^2 Y [dt + (x^2 + 2ms_1^2)d\sigma]^2}{(x^2 - y^2)(\gamma + y^2)H_1^2} - U \left(dt + y^2 d\sigma + \frac{(x^2 - y^2)fH_1 [abd\sigma + (\gamma + y^2)d\chi]}{ab(x^2 - y^2)H_3 - 2ms_3c_3(\gamma + y^2)} \right)^2 \Big\}, \quad (2.107)$$

$$\begin{aligned} \mathcal{A}^1 &= \mathcal{A}^2 = \frac{2ms_1c_1(dt + y^2d\sigma)}{(x^2 - y^2)H_1}, \\ \mathcal{A}^3 &= \frac{2m \{s_3c_3(dt + y^2d\sigma) - (s_1^2 - s_3^2)[abd\sigma + (\gamma + y^2)d\chi]\}}{(x^2 - y^2)H_3}, \end{aligned} \quad (2.108)$$

$$X_1 = X_2 = \left(\frac{H_3}{H_1}\right)^{1/3}, \quad X_3 = \left(\frac{H_1}{H_3}\right)^{2/3}, \quad (2.109)$$

with

$$\begin{aligned} f &= x^2 + \gamma + 2ms_3^2, \quad \gamma = 2abs_3c_3 + (a^2 + b^2)s_3^2, \\ U &= \frac{[ab(x^2 - y^2)H_3 - 2ms_3c_3(\gamma + y^2)]^2}{(x^2 - y^2)^2(\gamma + y^2)fH_1^2H_3}, \\ H_1 &= 1 + \frac{2ms_1^2}{x^2 - y^2}, \quad H_3 = 1 + \frac{2ms_3^2}{x^2 - y^2}, \end{aligned} \quad (2.110)$$

$$\begin{aligned} X &= \frac{-2mx^2 + (\tilde{a}^2 + x^2)(\tilde{b}^2 + x^2)}{x^2} \\ &\quad + \frac{g^2(\tilde{a}^2 + 2ms_1^2 + x^2)(\tilde{b}^2 + 2ms_1^2 + x^2)(2ms_3^2 + \gamma + x^2)}{x^2}, \\ Y &= \frac{(\tilde{a}^2 + y^2)(\tilde{b}^2 + y^2)[1 + g^2(\gamma + y^2)]}{y^2}, \end{aligned} \quad (2.111)$$

and

$$s_i = \sinh \delta_i, \quad c_i = \cosh \delta_i, \quad \tilde{a} = ac_3 + bs_3, \quad \tilde{b} = bc_3 + as_3. \quad (2.112)$$

The solution is characterized by six parameters: the cosmological constant $\Lambda = -6g^2$, the mass parameter m , the two rotation parameters a and b , and the two charge parameters δ_1 and δ_3 . It is evident from (2.108) that the charges carried by the gauge fields \mathcal{A}^1 and \mathcal{A}^2 are equal, whilst that carried by \mathcal{A}^3 is an independent parameter.

The solution can be rewritten in an asymptotically non-rotating frame, in terms of a canonically-normalized time coordinate τ and azimuthal coordinates ϕ and ψ

having independent periods 2π by means of the transformation

$$\begin{aligned} t &= \frac{(1 + g^2\gamma)\tau}{\Xi_a\Xi_b} - \frac{a(a^2 + \gamma)\phi}{(a^2 - b^2)\Xi_a} + \frac{b(b^2 + \gamma)\psi}{(a^2 - b^2)\Xi_b}, \\ \sigma &= \frac{g^2\tau}{\Xi_a\Xi_b} - \frac{a\phi}{(a^2 - b^2)\Xi_a} + \frac{b\psi}{(a^2 - b^2)\Xi_b}, \\ \chi &= \frac{g^4ab\tau}{\Xi_a\Xi_b} - \frac{b\phi}{(a^2 - b^2)\Xi_a} + \frac{a\psi}{(a^2 - b^2)\Xi_b}, \end{aligned} \quad (2.113)$$

with $\Xi_a = 1 - g^2a^2$ and $\Xi_b = 1 - g^2b^2$. It is also useful to defined new coordinates r and θ to replace x and y ,

$$\begin{aligned} x^2 &= r^2 - \gamma - \frac{2}{3}m(2s_1^2 + s_3^2), \\ y^2 &= -\tilde{a}^2 \cos^2 \theta - \tilde{b}^2 \sin^2 \theta = -\gamma - a^2 \cos^2 \theta - b^2 \sin^2 \theta. \end{aligned} \quad (2.114)$$

For later convenience, one can define a function $\Delta_r(r)$ by

$$r^2 \Delta_r(r) = x^2 X(x) \implies \frac{dx^2}{X(x)} = \frac{dr^2}{\Delta_r(r)}. \quad (2.115)$$

After rewriting the full metric in terms of these new coordinates, it can be seen that it describes a rotating black hole with an horizon of S^3 topology located at the largest root of the function $\Delta_r(r)$. At large distance, $r \rightarrow \infty$, the metric approaches anti-de Sitter spacetime ($R_{\mu\nu} \rightarrow -4g^2g_{\mu\nu}$),

$$\begin{aligned} ds^2 &\approx -\frac{(1 + g^2r^2)\Delta_\theta}{\Xi_a\Xi_b} d\tau^2 + \frac{dr^2}{g^2r^2} + \frac{\rho^2 d\theta^2}{\Delta_\theta} \\ &\quad + \frac{r^2 + a^2}{\Xi_a} \sin^2 \theta d\phi^2 + \frac{r^2 + b^2}{\Xi_b} \cos^2 \theta d\psi^2, \\ \rho^2 &= r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta, \\ \Delta_\theta &= 1 - g^2a^2 \cos^2 \theta - g^2b^2 \sin^2 \theta. \end{aligned} \quad (2.116)$$

A discussion of the thermodynamical properties of the solution can be found in [21]. One point to notice is that the mass is calculated by integrating the first law of

thermodynamics [115],

$$dE = TdS + \Omega_\phi dJ_\phi + \Omega_\psi dJ_\psi + \sum_i \Phi_i dQ_i. \quad (2.117)$$

The fact the right hand side of the above equation is an exact differential [21] not only provides a useful check on the algebra, but also indicates that the mass is well defined in this case.

A BPS limit of the non-extremal solutions can be obtained if the conserved charges satisfy the condition

$$E = gJ_\phi + gJ_\psi + \sum_{i=1}^3 Q_i. \quad (2.118)$$

Equivalent BPS conditions arise for all other choices of signs in this equation. The solution then admits a Killing spinor, implying that it is a supersymmetric supergravity background. Using the thermodynamical quantities given in [21], one finds from (2.118),

$$e^{2\delta_1+2\delta_3} = 1 + \frac{2}{g(a+b)}. \quad (2.119)$$

The existence of a Killing spinor η allows one to write down an everywhere-timelike Killing vector $K^\mu = \bar{\eta}\Gamma^\mu\eta$. This will take the form

$$K = \frac{\partial}{\partial\tau} + g\frac{\partial}{\partial\phi} + g\frac{\partial}{\partial\psi}. \quad (2.120)$$

Because it admits a spinorial square root, the Killing vector K has a manifestly negative norm (see, for example, [108], and also [102]), and in fact one can show that when (2.119) is satisfied

$$K^2 = -h_1^{-4/3}h_3^{-2/3}\left\{\rho^2 - \frac{2m}{3g^2(a+b)^2e^{2\delta_3}} + \frac{m[(2+ga+gb)^2 - g^2(a+b)^2e^{4\delta_3}](3\Delta_\theta - (1+ga)(1+gb))}{6g^2(a+b)^2(1+ga)(1+gb)(2+ga+gb)e^{2\delta_3}}\right\}^2,$$

$$(2.121)$$

$$h_1 = \rho^2 + \frac{2}{3}m(s_1^2 - s_3^2), \quad h_3 = \rho^2 - \frac{4}{3}m(s_1^2 - s_3^2). \quad (2.122)$$

This result is useful for studying the occurrence of closed timelike curves (CTCs) in the BPS metric. First, we note that the metric can be cast in the form

$$ds^2 = -\frac{r^2 \Delta_r(r) \Delta_\theta \sin^2 \theta \cos^2 \theta dt^2}{\Xi_a^2 \Xi_b^2 B_\phi B_\psi} + h_1^{2/3} h_3^{1/3} \left[\frac{d\theta^2}{\Delta_\theta} + \frac{dr^2}{\Delta_r(r)} \right] + B_\psi (d\psi + v_1 d\phi + v_2 dt)^2 + B_\phi (d\phi + v_3 dt)^2, \quad (2.123)$$

where the functions B_ϕ , B_ψ and v_i can be read off by comparing (2.123) with the original form of the metric. In order not to have CTCs, it must be that B_ϕ and B_ψ are non-negative outside the horizon. After imposing (2.119), we can write

$$K^2 = -\frac{r^2 \Delta_r(r) \Delta_\theta \sin^2 \theta \cos^2 \theta}{\Xi_a^2 \Xi_b^2 B_\phi B_\psi} + B_\psi (g + v_1 g + v_2)^2 + B_\phi (g + v_3)^2, \quad (2.124)$$

and so on the horizon, where $\Delta_r(r) = 0$, the negativity of K^2 implies that B_ϕ or B_ψ must be negative, and hence except for special cases there will be CTCs on and outside the horizon in the BPS solutions.

One way to avoid the occurrence of CTCs outside the horizon in the BPS solutions is to let K^2 vanish on the horizon. As in cases studied previously (see, e.g. [102]), this condition is precisely equivalent to

$$\Delta_r(r_0) = 0 = \Delta'_r(r_0), \quad (2.125)$$

which also means that the Hawking temperature vanishes. This is indeed a necessary condition for having a regular supersymmetric black hole, since the inequivalent energy distribution functions for bosons and fermions in a thermal state at non-zero temperature are manifestly incompatible with supersymmetry.

A convenient way to solve the zero-temperature condition (2.125) in addition to

the BPS condition (2.119) is to regard (2.119) as placing a constraint on the value of the gauge-coupling constant g as a function of the rotation and charge parameters. (This has the advantage of allowing not only the two angular momenta, but also the two charge parameters, to be adjusted freely, and this makes it easier to compare results with previously-known cases such as $\delta_1 = \delta_3$, $\delta_1 = 0$ or $\delta_3 = 0$.) The zero-temperature condition (2.125) can then be solved for the mass parameter m , implying that

$$M = \frac{e^{\delta_1 + \delta_3} [(a^2 + b^2) \sinh(2\delta_1 + 2\delta_3) + 2ab \cosh(2\delta_1 + 2\delta_3)]}{2 \sinh(\delta_1 + \delta_3) \sinh 2\delta_1}. \quad (2.126)$$

If the solution is chosen so that both (2.119) and (2.126) are satisfied, then it can describe a regular supersymmetric black hole. It is still necessary to restrict the remaining 3 parameters to lie within appropriate regions, in order that the metric be free of any CTCs outside the horizon, but these remaining conditions take the form of inequalities rather than further functional relations between the parameters. They are generalizations of the restrictions found in [102] for the case when the three charges were equal. One can, for example, see that if ga and gb are sufficiently small and positive, and the charge parameter δ_3 is sufficiently large, then there will be no CTCs outside the horizon. The supersymmetric black holes that we have obtained here will correspond to the $Q_1 = Q_2$ specialization of the supersymmetric 3-charge black holes constructed in [94].

A second way of eliminating CTCs in the BPS solutions is if the product $B_\phi B_\psi$ is proportional to $\Delta_r(r)$, and hence one or other of B_ϕ or B_ψ vanishes on the horizon. In this case, the BPS condition (2.119) is supplemented by the further condition

$$m = \frac{2k_3 k_4 (a + b)(1 + ga)(1 + gb)(2 + ga + gb)e^{2\delta_3}}{k_1^2 k_2}, \quad (2.127)$$

with

$$k_1 = (2 + ga + gb)^2 - g^2(a + b)^2 e^{4\delta_3}, \quad (2.128)$$

$$k_2 = (2 + ga + gb)(a + b + 2gab) - (a + b)(2 + ga + gb + 2g^2ab)e^{4\delta_3},$$

$$k_3 = (2 + ga + gb) [2a - gb^2 + gab(1 - ga - gb)] + g(a + b)^2(2 + gb + g^2ab)e^{4\delta_3},$$

$$k_4 = (2 + ga + gb) [2b - ga^2 + gab(1 - ga - gb)] + g(a + b)^2(2 + ga + g^2ab)e^{4\delta_3}.$$

In this case, we have chosen to use (2.119) to eliminate δ_1 . The metric now describes a smooth topological soliton, with $r = r_0$ being a regular origin of polar coordinates at which $B_\phi \rightarrow 0$, and free of conical singularities, provided that the quantization condition

$$\frac{ak_4 - bk_3}{g(a - b)b(1 - ga)} \left[\frac{2g^2b}{k_1} - \frac{1 + gb}{k_2} - \frac{g(a - b)(1 - gb)}{k_4} \right] = 1 \quad (2.129)$$

is satisfied. These topological solitons generalize examples found in [102] in the case that the three charges were equal.

F. Summary of the Chapter

In this chapter, we have constructed two set of solutions in two different theories. The first is a Plebanski-Demianski type solution (2.45) in five dimensions. The solution is then generalized to include electric and magnetic dipole charges (2.89). The second is a three-charge (two of which equal) two-rotation solution (2.107) to the five dimensional maximal supergravity.

In the first case, the neutral solution (2.45) has three non-trivial (dimensionless) parameters, which is one more than the number in the rotating black hole solution in the same dimension. We identified three limiting cases that are of particular interest. The first is a limit that gives back the standard rotating black hole [71],

with two independent rotation parameters. The second is a limit that gives the original single-rotation black ring, which was found in [89]. The last is a limit giving rise to a new family of static metrics, with two non-trivial dimensionless parameters. For this last limit, we find that the requirement of no conical singularities imposes periodicity conditions on the azimuthal coordinates which imply that the horizon has the topology of the lens space $L(n; m)$, where m and n are positive integers satisfying $m \geq n + 2 \geq 3$. The lens space $L(n; m)$ is defined as a factoring S^3 by a certain freely-acting discrete subgroup $\Gamma(n; m)$ of the $SO(4)$ isometry group. The black hole horizon is an inhomogeneous distortion of the “round” lens space. By contrast, asymptotically at infinity the spacetime approaches $(\text{Minkowski})_5/\Gamma(m; n)$. This means that the spatial sections at large radius are lens spaces $L(m; n)$. We calculated all the conserved charges and thermodynamic quantities for these lens-space black holes, and showed that the first law of thermodynamics is satisfied. Our solutions demonstrate that black holes with $(\text{Minkowski})_5/\Gamma(m; n)$ asymptotic structure and a given mass are not unique.

We have also investigated the global structure of the general neutral solution with three non-trivial dimensionless parameters. We find that (except for limiting cases that reduce to the previous discussion) the avoidance of conical singularities now requires that the time coordinate also be identified periodically. This is reminiscent of the situation in the Taub-NUT metrics in four dimensions. In fact, one can take the view that the general new neutral solution are the five-dimensional analogue of the four-dimensional rotating Taub-NUT metrics. In contrast, the general construction of higher-dimensional rotating Taub-NUT metrics in [107, 88] gave only a trivial “NUT parameter” in the special case of five dimensions.

The charged version of the solution (2.89) carries both electric charge and magnetic dipole charge. A special limit of the solution gives back the charged rotating

black hole in five-dimensional minimal supergravity. In another limit, the solution reduces to a class of black rings found in [113, 114]. For the global properties of the general solution (2.89), we find that provided the parameters are chosen properly, with two algebraic conditions characterized by coprime integers m and n , the solution describes a stationary black hole spacetime that is asymptotically locally flat, with an horizon that is topologically the lens space $L(n; m) = S^3/\Gamma(n; m)$. At large distance, the spacetime approaches $(\text{Minkowski})_5/\Gamma(m; n)$. The two algebraic conditions involving m and n arose from requiring that the spacetime be free from conical singularities.

In the second case, we consider solutions in the five dimensional maximal supergravity. The most general non-extremal black holes with an S^3 horizon topology in maximal $SO(6)$ -gauged five-dimensional supergravity would be characterized by a total of six parameters, comprising the mass, the two independent angular momenta, and three independent electric charges supported by the three abelian gauge fields in the $U(1)^3$ Cartan subgroup of $SO(6)$. They could equivalently be regarded as solutions in $\mathcal{N} = 2$ gauged supergravity coupled to two vector multiplets.

Here we have constructed the most general non-extremal rotating black holes in the five dimensional maximal supergravity to date. They are characterized by five parameters, namely the mass, the two angular momenta, and two independently-specifiable charge parameters. They correspond to the situation where two of the three charges in the most general solution are set equal, but with no restrictions otherwise. These solutions encompass and extend all previously-obtained results for black holes with independent rotation parameters in five-dimensional gauged supergravity.

We calculated the conserved angular momenta and charges for the new solutions, the entropy and Hawking temperature, and the angular velocities and electric potentials on the horizon. From this, we showed that the first law of thermodynamics is

integrable, and we obtained the expression for the mass of the black holes. We then studied the BPS limit of the solutions, and showed how further restrictions on the remaining parameters would give rise to regular supersymmetric black holes and to smooth topological solitons.

The results we have obtained in this paper should have applications in the study of the AdS/CFT correspondence. It would be of considerable interest to find the more general 6-parameter black-hole solutions in five-dimensional maximal gauged supergravity, in which the three electric charges, as well as the mass and the two angular momenta, are independently specifiable. These can be expected to be considerably more complicated than the solutions constructed until now.

CHAPTER III

COUNTING THE BLACK HOLE ENTROPY*

The most apparent reason to assign an entropy to a black hole is the need to keep the second law of thermodynamics alive. If the black holes did not have an entropy, then one can reduce the entropy of the Universe by simply dropping matter into a black hole. But finally, it was the observation that the horizon area of black holes never decrease in any natural process [48] that helped Bekenstein to suggest that the entropy of a black hole is proportional to its horizon area [6]. Bekenstein proposed the generalized second law of thermodynamics, that the total entropy of the Universe, including that from black holes, should never decrease. The thermodynamics of black holes was established when the first four laws were formulated by Bardeen, Carter and Hawking in 1973 [7], and when Hawking found that black holes radiate at finite temperatures [8]. The need to give a microscopic explanation for the black hole entropy and the search for a quantum theory of gravity have combined to become a research topic that have last for dozens of years.

The first breakthrough was made in 1996 when Strominger and Vafa calculated the entropy for a certain type of black holes using string theory [20]. When the dust was settled, however, it was realized that neither string theory nor supersymmetry was crucial in the calculation of the entropy [53]. What's important is some conformal

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symmetries related to the black hole horizon [52]. According to Boltzmann, the entropy is related to the number of degeneracies for a given classical system. For a black hole, one should then look for the degeneracies on the horizon. This is because any change outside of the horizon will lead to a different black hole, while changes inside the horizon are believed to be irrelevant to the outside world. In this respect, the latest development is the Kerr/CFT correspondence proposed by Strominger and collaborators [26]. In their work, they zoom in on the metric near the horizon, and then study possible degenerate configurations of the near-horizon metric. With the help of appropriate boundary conditions, some particular phase space is singled out as containing the desired configurations. The phase space is then identified with that of a conformal field theory, and the entropy is calculated from the corresponding central charge by using Cardy's formula [61].

In this process, it is of crucial importance to be able to identify the phase space with that of a conformal field theory. This is done by identifying symmetries of the phase spaces. The symmetries on the gravity side are asymptotic symmetries defined with appropriate boundary conditions. In this respect, the problem is related to the construction of conservation laws in a curved spacetime, which involves exact symmetries. For the latter case, the most important problem is a general proof of the first law of thermodynamics for black holes [7, 116, 117]. Notably, there is no known formulae that can give the total energy in an AdS background for a system in arbitrary dimensions. For the commonly used methods, the Arnowitt-Deser-Misner (ADM) method [42] and the Komar formulae [118, 119] work best in the Minkowski background, the Ashtekar-Magnon-Das (AMD) method [120, 121] and its variations [122] do not work for dimensions lower than four, and the Abbott-Deser method [123] and the boundary subtraction method are marred with ambiguities [124]. A comparison of various methods and more references can be found in [122].

So to get a better idea of the whole picture, we will devote the first section of this chapter to discussing the formulation of conservation laws in a curved spacetime. The formalism needed for the Kerr/CFT calculation will also be developed in this process. In the second section, we will then show that the Kerr/CFT correspondence can be used to calculate the entropy for all known extremal stationary and axisymmetric black holes. This is done with the help of two ansätze that are general enough to cover all known black hole solutions. The main result of this chapter has been submitted for publication in [125].

A. Conservation Laws in General Relativity

In this chapter, we will discuss the construction of conservation laws in curved spacetimes. Apart from the familiar notion that one needs the presence of spacetime symmetries (isometries characterized by Killing vectors) to get conservation laws, we will show that equations of motion for secondary fields (to be defined below (3.32)) also play a crucial role in this process. We will also explain the treatment of asymptotic symmetries by using the covariant phase space method. Numerous examples (such as the construction of the first law of thermodynamics for black holes in simple systems) will be used to illustrate the basic ideas. Most of the material involved in this section is not new. In fact, we will closely follow the treatment in existing works such as [116, 117, 126, 127] in various places. However, by going through the whole exercise, we wish to get an organized understanding of the topics involved. Results needed in the calculation of the Kerr/CFT correspondence will also be developed in the process.

1. Noether's Theorem and Its Limitation

To warm up, let's start with a system defined in a flat spacetime and with a Lagrangian containing no more than the first order derivative on the fields. The Lagrangian can be schematically written as $\mathcal{L} = \mathcal{L}(\phi^a, \partial_\mu \phi^a)$, and the action

$$S = \int d^n x \mathcal{L}(\phi^a, \partial_\mu \phi^a). \quad (3.1)$$

Note we will be using the most negative metric $\eta = \text{diag}\{1, -1, \dots, -1\}$ through out this subsection. The classical field configuration is obtained by enforcing the action principle, i.e., if one varies the fields around their classical value but keeps $\delta\phi^a = 0$ on the boundary, then $\delta S = 0$. From

$$\delta S = \int d^n x \left\{ \left[\frac{\partial \mathcal{L}}{\partial \phi^a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^a)} \right) \right] \delta \phi^a + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^a)} \delta \phi^a \right) \right\}, \quad (3.2)$$

the field equations can be read off as

$$\frac{\delta \mathcal{L}}{\delta \phi^a} = \frac{\partial \mathcal{L}}{\partial \phi^a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^a)} \right) = 0. \quad (3.3)$$

This is called the Euler-Lagrangian equation. For theories with up to the k -th order derivatives on fields, one can generalize the Euler-Lagrange equation by writing

$$\frac{\delta \mathcal{L}}{\delta \phi^a} = \frac{\partial \mathcal{L}}{\partial \phi^a} + \sum_{i=1}^k (-)^i \partial_{\mu_1 \dots \mu_i} \left(\frac{\partial \mathcal{L}}{\partial (\partial_{\mu_1 \dots \mu_i} \phi^a)} \right) = 0, \quad (3.4)$$

where $\partial_{\mu_1 \dots \mu_i} = \partial_{\mu_1} \dots \partial_{\mu_i}$.

Now consider a transformation of the coordinates and fields,

$$\delta x^\mu = x'^\mu - x^\mu, \quad \delta \phi(x) = \phi'(x') - \phi(x). \quad (3.5)$$

One gets

$$d^n x' = (1 + \partial_\mu \delta x^\mu) d^n x, \quad (3.6)$$

and

$$\begin{aligned}
\delta S &= \int d^n x \left(\partial_\mu \delta x^\mu \mathcal{L} + \delta \mathcal{L} \right) = \int d^n x \left[\partial_\mu (\delta x^\mu \mathcal{L}) + \bar{\delta} \mathcal{L} \right] \\
&= \int d^n x \left[\partial_\mu (\delta x^\mu \mathcal{L}) + \frac{\partial \mathcal{L}}{\partial \phi^a} \bar{\delta} \phi^a + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^a)} \bar{\delta} (\partial_\mu \phi^a) \right] \\
&= \int d^n x \partial_\mu \left[\delta x^\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^a)} \bar{\delta} \phi^a \right] \\
&\quad + \int d^n x \left[\frac{\partial \mathcal{L}}{\partial \phi^a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^a)} \right) \right] \bar{\delta} \phi^a, \tag{3.7}
\end{aligned}$$

where

$$\bar{\delta} f(x) \equiv f'(x) - f(x) \implies \bar{\delta} (\partial_\mu f) = \partial_\mu (\bar{\delta} f). \tag{3.8}$$

When (3.5) is a symmetric transformation of the system, one gets a conserved Noether current, $\delta S = 0 \implies \partial_\mu \mathcal{J}^\mu = 0$,

$$\begin{aligned}
\mathcal{J}^\mu &= - \left[\delta x^\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^a)} \bar{\delta} \phi^a \right] \\
&= \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^a)} \partial_\nu \phi^a - \eta_\nu^\mu \mathcal{L} \right] \delta x^\nu - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^a)} \delta \phi^a, \tag{3.9}
\end{aligned}$$

where we have applied (3.3). Note that

$$\mathcal{J}^0 = [\pi_a \partial_\nu \phi^a - \eta_\nu^0 \mathcal{L}] \delta x^\nu - \pi_a \delta \phi^a, \quad \pi_a = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^a}, \quad \dot{\phi}^a = \partial_t \phi^a. \tag{3.10}$$

The nature of the charge will depend on the corresponding symmetry.

A symmetric transformation is always characterized by some gauge parameter. Since the value of a gauge parameter is *a priori* arbitrary, any physically meaningful charges should be independent of it. This goal is easy to achieve when the gauge parameters are constant (which means that the symmetry is global). In this case, the Noether current must be of the form

$$\mathcal{J}_\epsilon^\mu = \epsilon^\alpha \cdot \mathcal{J}_\alpha^\mu, \tag{3.11}$$

where \mathcal{J}_α^μ is independent of the gauge parameters ϵ^α , and the index α labels different symmetries. Given appropriate boundary conditions, a conserved charge corresponding to ϵ^α can be well defined,

$$Q_\alpha = \int_V d^{n-1}\mathbf{x} \mathcal{J}_\alpha^0. \quad (3.12)$$

In the case when the gauge parameters are spacetime dependent (which means that the symmetry is local), one can still try to use a similar integral to define the charge. However, such an integral will in general contain the arbitrary gauge parameter, and so the resulted charge cannot be physically meaningful.

For local symmetries, one expects to find the corresponding charges as well. But it is obvious that one cannot get it from the construction above. To figure out how to fix the problem, let's look at a more general system. Partially following [126, 127], we now assume that the Lagrangian can contain arbitrary number of derivatives on the fields, and that the spacetime can be curved as well. An infinitesimal variation of the action is

$$\delta S = \int d^n x \left(\frac{\delta \mathcal{L}}{\delta \phi^a} \delta \phi^a - \partial_\mu \mathcal{J}^\mu \right), \quad (3.13)$$

where by using the partial integration repeatedly, we have moved all terms with a derivative on $\delta \phi^a$ into the \mathcal{J}^μ term. It is easy to tell that all other terms combine to become the Euler-Lagrange derivatives. For a gauge symmetry,

$$\frac{\delta \mathcal{L}}{\delta \phi^a} \delta_\epsilon \phi^a - \partial_\mu \mathcal{J}_\epsilon^\mu = 0. \quad (3.14)$$

When the gauge parameter ϵ is constant, both $\delta_\epsilon \phi^a$ and $\partial_\mu \mathcal{J}_\epsilon^\mu$ are proportional to ϵ . So one can solve \mathcal{J}^μ from (3.14), and then use it to define a charge as in (3.11) and (3.12). When the ϵ is spacetime dependent, $\delta_\epsilon \phi^a$ will involve terms proportional to ϵ and its derivatives. In this case, we can apply the partial integration repeatedly to

write

$$\frac{\delta \mathcal{L}}{\delta \phi^a} \delta_\epsilon \phi^a = \epsilon \cdot \delta^+(\phi, \frac{\delta \mathcal{L}}{\delta \phi}) + \partial_\mu S_\epsilon^\mu(\phi, \frac{\delta \mathcal{L}}{\delta \phi}), \quad (3.15)$$

where all the terms involving a derivative on ϵ are moved into the S_ϵ^μ term. At any chosen spacetime point, the derivatives of ϵ can always be made independent of ϵ itself. So as long as (3.14) holds, one has

$$\delta^+(\phi, \frac{\delta \mathcal{L}}{\delta \phi}) = 0, \quad \mathcal{J}_\epsilon^\mu = S_\epsilon^\mu(\phi, \frac{\delta \mathcal{L}}{\delta \phi}) + \partial_\nu f_\epsilon^{\mu\nu}, \quad (3.16)$$

where $f_\epsilon^{\mu\nu}$ is an arbitrary antisymmetric tensor, and so it is trivial as far as physics is concerned. We shall always let $f_\epsilon^{\mu\nu} = 0$. Note that S_ϵ^μ only contains terms proportional to $\frac{\delta \mathcal{L}}{\delta \phi}$ and its derivatives, and so the whole function vanishes when (3.4) is satisfied. As a result, (3.16) means that the Noether current vanishes on shell if the gauge parameters are spacetime dependent.

Now we have got two problems. The first is that when the Noether current \mathcal{J}_ϵ^μ vanishes on shell, the information about the symmetry seems to have got lost. The second is that, even if the current \mathcal{J}_ϵ^μ in (3.16) did not vanish on shell, it is not going to give any physically sensible charge. This is because the \mathcal{J}_ϵ^μ depends on the arbitrary gauge parameter, which cannot be pulled out of the charge integral in general. But there is no reason to avoid using this current, because it *is* a conserved current by construction. The second problem is actually solved by what's causing the first problem, i.e., by $\mathcal{J}_\epsilon^\mu = 0$. In this case, if one expands \mathcal{J}_ϵ^μ in terms of the gauge parameter and its derivatives, all the expansion coefficients will be zero. Now it is easy to imagine that all the information about the symmetry is contained in the extra constraints that could have resulted from the vanishing of the expansion coefficients. However, (3.16) tells us that one can get $\mathcal{J}_\epsilon^\mu = 0$ by using the equations of motion (3.4) alone. So those “extra” constraints are nothing but the equations of

motion themselves. As a result, the equations of motion also contain all the symmetry information that one is looking for.

In the following we shall use examples to explain some of the points in more detail.

The first example is the translation symmetry for a system described by (3.1),

$$x \longrightarrow x' = x + a \implies \delta x^\mu = a^\mu, \quad \delta \phi^a = 0, \quad (3.17)$$

$$\implies \mathcal{J}^0 = \mathcal{H} a^0 + \pi_a \nabla \phi^a \cdot \mathbf{a}, \quad \mathcal{H} = \pi_a \dot{\phi}_i - \mathcal{L}. \quad (3.18)$$

Corresponding to $\delta t = a^0$ and $\delta \mathbf{x} = \mathbf{a}$, we have from (3.12),

$$H = \int_{\Sigma_t} d^{n-1} \mathbf{x} \mathcal{H} = \int_{\Sigma_t} d^{n-1} \mathbf{x} (\pi_a \dot{\phi}_i - \mathcal{L}), \quad p_i = \int_{\Sigma_t} d^{n-1} \mathbf{x} \pi_a \partial_i \phi^a. \quad (3.19)$$

Note (3.12) can be used only because the gauge parameters a^0 and \mathbf{a} are constant.

A second example is the Klein-Gordon theory of a complex scalar field,

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi. \quad (3.20)$$

The theory has a global $U(1)$ gauge symmetry,

$$\phi \longrightarrow \phi' = e^{i\alpha} \phi \implies \delta \phi = i\alpha \phi. \quad (3.21)$$

From (3.9),

$$\mathcal{J}^\mu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \delta \phi^* - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta \phi = i\alpha(\phi^* \partial^\mu \phi - \partial^\mu \phi^* \phi). \quad (3.22)$$

The charge (3.12) is

$$Q = \int_{\Sigma_t} d^{n-1} \mathbf{x} i(\phi^* \partial_t \phi - \partial_t \phi^* \phi). \quad (3.23)$$

Again, this charge is possible because the gauge parameter α is constant.

A third example is the quantum electrodynamics (QED),

$$\mathcal{L} = i\bar{\psi}\gamma^\mu(\partial_\mu + igA_\mu)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - m\bar{\psi}\psi, \quad (3.24)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and g is the charge of the fermion. For an electron, one has $g = -e \approx -1.6 \times 10^{-19}C$. The equations of motion are

$$g\bar{\psi}\gamma^\mu\psi + \partial_\nu F^{\mu\nu} = 0, \quad (3.25)$$

$$i\gamma^\mu(\partial_\mu + igA_\mu)\psi - m\psi = 0. \quad (3.26)$$

The system has a local $U(1)$ gauge symmetry,

$$\begin{cases} \psi & \longrightarrow & e^{i\alpha}\psi & : & \delta\psi = i\alpha\psi, \\ A_\mu & \longrightarrow & A_\mu - \frac{1}{g}\partial_\mu\alpha & : & \delta A_\mu = -\frac{1}{g}\partial_\mu\alpha. \end{cases} \quad (3.27)$$

Using (3.9), one has

$$\begin{aligned} \mathcal{J}^\mu &= -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi - \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\delta A_\nu \\ &= -i\bar{\psi}\gamma^\mu(i\alpha\psi) - (-F^{\mu\nu})\left(-\frac{1}{g}\partial_\nu\alpha\right) \\ &= \frac{\alpha}{g}(g\bar{\psi}\gamma^\mu\psi + \partial_\nu F^{\mu\nu}) - \frac{1}{g}\partial_\nu(\alpha F^{\mu\nu}). \end{aligned} \quad (3.28)$$

In the last line, the second term is trivial since it automatically drops out of the integral $\int d^n x \partial_\mu \mathcal{J}^\mu$. The first term vanishes as well due to (3.25). So the the Noether current vanishes on shell, just as said bellow (3.16).

To recover the information about the symmetry, we need to look at the equations of motion. Since (3.25) is the one responsible for the vanishing of \mathcal{J}^μ in (3.28), it should also be the one containing the symmetry information. When integrated over a space-like hypersurface, one has from (3.25),

$$0 = \int_V d^3\mathbf{x} \left(g\bar{\psi}\gamma^0\psi + \partial_\nu F^{0\nu} \right). \quad (3.29)$$

Using $\mathbf{E} = -\nabla\phi - \partial_t\mathbf{A}$ and $A^\mu = \{\phi, \mathbf{A}\}$, one has

$$\partial_\nu F^{0\nu} = \partial_i F^{0i} = \partial_i(\partial^0 A^i - \partial^i A^0) = \partial_i(\partial_0 A^i + \partial_i A^0) = -\nabla \cdot \mathbf{E}. \quad (3.30)$$

On the other hand, it can be shown that [2]

$$\begin{aligned} g \int_V d^3\mathbf{x} : \bar{\psi} \gamma^0 \psi : &= g \int_V d^3\mathbf{x} : \psi^\dagger \psi : = g \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_s \left(a_p^{s\dagger} a_p^s + b_p^s b_p^{s\dagger} : \right) \\ &= g \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_s \left(a_p^{s\dagger} a_p^s - b_p^{s\dagger} b_p^s \right), \end{aligned} \quad (3.31)$$

which is the net charge of the fermions. So (3.29) is just the usual Gauss's law, familiar in electromagnetics,

$$g \int_V d^3\mathbf{x} \psi^\dagger \psi = Q_g = \int_V d^3\mathbf{x} \nabla \cdot \mathbf{E} = - \oint_{\partial\Sigma_t} d^2 S_i F^{0i}. \quad (3.32)$$

This equation also defines a physically meaningful charge.

To generalize to other cases, such as gravity, one may notice that (3.25) are equations of motion for the gauge field. The gauge field can be viewed a secondary field in the sense that it cannot have a nontrivial static configuration unless there exists other matter field as a source. So our conclusion from this exercise is that, in cases when the Noether current vanishes, one should always try to use the equations of motion (for secondary fields) to construct the corresponding conservation laws.

On the other hand, the problem with using the equations of motion is that one does not have a formulae for the charge directly. What's worse, one often has more than one ways to write a same set of equations of motion. As a result, one has to use other information (such as the expected physical interpretation) to define a physically meaningful charge. We will see this explicitly in the next subsection.

2. Exact Symmetries in General Relativity

Before discussing symmetries in a curved spacetime, it is important to distinguish between the diffeomorphism invariance and the spacetime symmetry (isometry). The diffeomorphism invariance merely says that one can label the spacetime points in whichever way he likes. For a general and infinitesimal coordinate transformation, $\epsilon^\mu(x) = x'^\mu - x^\mu \rightarrow 0$, one has

$$\begin{aligned} g_{\mu\nu} &= \partial_\mu x'^\alpha \partial_\nu x'^\beta g'_{\alpha\beta} \approx g'_{\mu\nu} + g_{\alpha\nu} \partial_\mu \epsilon^\alpha + g_{\mu\beta} \partial_\nu \epsilon^\beta \\ &= g'_{\mu\nu} - \epsilon^\alpha \partial_\alpha g_{\mu\nu} + \mathcal{L}_\epsilon g_{\mu\nu}, \\ \implies \delta g_{\mu\nu} &= g'_{\mu\nu} - g_{\mu\nu} = \epsilon^\alpha \partial_\alpha g_{\mu\nu} - \mathcal{L}_\epsilon g_{\mu\nu}, \end{aligned} \quad (3.33)$$

where the Lie derivative \mathcal{L}_ϵ is defined in (A.15). The spacetime symmetry describes symmetry properties of the spacetime geometry. When one moves along a vector ξ , the variation of the metric is given by $\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}$. A spacetime symmetry is characterized by a Killing vector, along which the metric is invariant $\mathcal{L}_\xi g_{\mu\nu} = 0$. Killing vectors play the role of generators of the symmetry group. Only the spacetime symmetries have direct links to conservation laws in a curved spacetime.

a. Charges via the Equations of Motion

In the paragraph following (3.16), we concluded that conservation laws can be constructed from the equations of motion for secondary fields. In curved spacetimes the metric plays the role of a secondary field, because it will not have any nontrivial static configurations unless there exist other matter fields.

To see how this can be done, let's consider the action

$$S = \int_{\mathcal{M}} d^n x \sqrt{|g|} \mathcal{L}, \quad \mathcal{L} = \frac{R}{16\pi} + \mathcal{L}_m, \quad (3.34)$$

where \mathcal{L}_m denotes the matter contribution. For an infinitesimal variation of the metric,

$$\delta(\sqrt{|g|} R) = \sqrt{|g|} \left(R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} - \nabla_\mu \nabla_\nu + g_{\mu\nu} \nabla_\rho \nabla^\rho \right) \delta g^{\mu\nu}. \quad (3.35)$$

The equations of motion for the metric are

$$E^{\mu\nu} = 8\pi T^{\mu\nu}, \quad T^{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta(\sqrt{|g|} \mathcal{L}_m)}{\delta g_{\mu\nu}}, \quad (3.36)$$

where $E^{\mu\nu} = R^{\mu\nu} - \frac{R}{2} g^{\mu\nu}$. Note that from (A.7),

$$\nabla_\mu E^{\mu\nu} = 0. \quad (3.37)$$

By analogy to (3.32), one can try to construct a conserved charge with the help of the equations of motion (3.36). However, since $E^{\mu\nu}$ has two symmetric indices, a conserved current can be constructed only with the help of a Killing vector,

$$\mathcal{J}_{\xi,E}^\mu = E^{\mu\nu} \xi_\nu \implies \nabla_\mu \mathcal{J}_{\xi,E}^\mu = \nabla_\mu (E^{\mu\nu} \xi_\nu) = E^{\mu\nu} \nabla_\mu \xi_\nu = 0. \quad (3.38)$$

Note that although this current is the most straight forward choice based on (3.36), it may not be the best one. The reason is that (3.36) can also be written as

$$R^{\mu\nu} = 8\pi \left(T^{\mu\nu} - \frac{T}{n-2} g^{\mu\nu} \right), \quad (3.39)$$

which suggests a current of the form $\mathcal{J}_{\xi,R}^\mu = R^{\mu\nu} \xi_\nu$. This current is also conserved because of (A.22). Apart from these, there are also other choices for the conserved current, such as $\tilde{\mathcal{J}}_{\xi,R}^\mu = R \xi^\mu$, which is conserved because of (A.23). All such currents are equally good in defining their corresponding charges.

To figure out which one is the best, one needs the help from some known physical relations. Here we will turn to the first law of thermodynamics, which is expected to be satisfied by black holes. Some basic properties of black hole metrics can be found

near (3.118) and (3.119). The construction here follows that in [7, 116], and charges will be given in terms of the Komar integrals [118].

In the case when the exterior of a black hole is Ricci flat, $R_{\mu\nu} = 0$, and one has

$$\begin{aligned} 0 &= \int_V (d^{n-1}x)_\mu R^\mu_\nu \xi^\nu = \int_V (d^{n-1}x)_\mu \nabla_\rho \nabla^\mu \xi^\rho \\ &= \oint_{\partial\Sigma_t} (d^{n-2}x)_{\mu\nu} \nabla^\mu \xi^\nu = \left(\oint_{+\infty} - \oint_H \right) (d^{n-2}x)_{\mu\nu} \nabla^\mu \xi^\nu. \end{aligned} \quad (3.40)$$

Here $\int_{+\infty}$ means integrating over the spatial infinity, while \int_H means integrating over the black hole horizon. On the horizon, one can write the measure as

$$(d^{n-2}x)_{\mu\nu} = \frac{1}{2}(\xi_\mu n_\nu - \xi_\nu n_\mu) dA, \quad (3.41)$$

where ξ is defined in (3.129), and n_ν is an auxiliary null vector on the horizon (normalized by $n_\mu \xi^\mu = 1$). Note that $\xi^\rho \nabla_\rho \xi^\mu = \kappa \xi^\mu$ with κ being the surface gravity. One can derive from (3.40) that

$$\begin{aligned} 0 &= \oint_{+\infty} (d^{n-2}x)_{\mu\nu} \nabla^\mu \xi^\nu - \kappa A = \frac{1}{2} \oint_{+\infty} *d\xi - \kappa A \\ &= \frac{1}{2} \oint_{+\infty} *dk + \frac{1}{2} \Omega_a \oint_{+\infty} *d\ell_a - \kappa A \\ &= 8\pi \left(\frac{n-3}{n-2} M - \Omega_a J_a - \frac{\kappa A}{8\pi} \right), \end{aligned} \quad (3.42)$$

where $k = \partial_t$, $\ell_a = \partial_{\phi^a}$, and

$$M = \frac{n-2}{16\pi(n-3)} \oint_{+\infty} *dk, \quad J_a = -\frac{1}{16\pi} \oint_{+\infty} *d\ell_a. \quad (3.43)$$

A nice discussion of the historical development of (3.42) in four dimensions ($n = 4$) can be found in [128].

Equation (3.42) is not very good in telling us about the physical meaning of the charges defined in (3.43). To get a better idea, let's look at an infinitesimal change in the spacetime geometry, in the case when the parameters in the solution are varied.

In this process, it is convenient to use a coordinate system in which the two metrics share the same set of Killing vectors. This is always possible because changing the parameters will not affect the symmetry properties of the solution. However, to get the same set of Killing vectors, it is necessary to use coordinates in which the metrics are static at the spatial infinity.³ When this is done, one has

$$\begin{aligned} \delta k^\mu = \delta \ell_a^\mu = 0 &\implies \delta \xi^\mu = \delta(k^\mu + \Omega_a \ell_a^\mu) = \delta \Omega_a \ell_a^\mu, \\ &\implies \delta \xi_\mu = h_{\mu\nu} \xi^\nu + \delta \Omega_a \ell_{a\mu}, \end{aligned} \quad (3.44)$$

where $h_{\mu\nu} = \delta g_{\mu\nu}$. The detailed calculation of varying (3.42) has been done in [7]. The result is

$$\delta M = \frac{\kappa \delta A}{8\pi} + \Omega_a \delta J_a, \quad (3.45)$$

which can also be derived by using dimensional analysis [116]. As is already known, κ is related to the temperature $T = \kappa/2\pi$ and A is related to the entropy $S = A/4$ [8]. So from (3.45),

$$\delta M = T \delta S + \Omega_a \delta J_a. \quad (3.46)$$

This can be viewed as the first law of thermodynamics for a rotating black hole. Now it is obvious that M should be identified with the total energy, Ω_a should be identified with the angular velocity along ϕ^a and J_a should be identified with the corresponding angular momentum.

We can repeat the same process for cases with matter contributions. As an example, let's look at the Einstein Maxwell system,

$$\mathcal{L} = \frac{R}{16\pi} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (3.47)$$

³In the case when the coordinates are rotating at the spatial infinity, the rotation will be different when the parameters in the solution are changed. As a result, Killing vectors built out of the azimuthal angles will also be different [129].

The equations of motion are

$$\frac{R_{\mu\nu}}{16\pi} = \frac{1}{2}F_{\mu\alpha}F_{\nu}^{\alpha} - \frac{F_{\alpha\beta}F^{\alpha\beta}}{4(n-2)}g_{\mu\nu}, \quad (3.48)$$

$$0 = \nabla_{\mu}F^{\mu\nu}. \quad (3.49)$$

Since the Ricci tensor is nonzero in general, (3.40) and (3.42) need to be modified,

$$\begin{aligned} & 8\pi \left(\frac{n-3}{n-2}M - \Omega_a J_a - \frac{\kappa A}{8\pi} \right) \\ &= \oint_{\partial\Sigma_t} (d^{n-2}x)_{\mu\nu} \nabla^{\mu} \xi^{\nu} = \int_V (d^{n-1}x)_{\mu} R_{\nu}^{\mu} \xi^{\nu} \\ &= \int_V (d^{n-1}x)_{\mu} \left[\frac{1}{2} F^{\mu\alpha} F_{\nu\alpha} \xi^{\nu} - \frac{F_{\alpha\beta} F^{\alpha\beta}}{4(n-2)} \xi^{\mu} \right] \\ &= \int_V (d^{n-1}x)_{\mu} \nabla_{\nu} \tau^{\mu\nu}, \end{aligned} \quad (3.50)$$

where

$$\tau^{\mu\nu} = -\frac{1}{2}F^{\mu\nu}A_{\alpha}\xi^{\alpha} + \frac{1}{2(n-2)}A_{\alpha}\left(F^{\mu\alpha}\xi^{\nu} - F^{\nu\alpha}\xi^{\mu}\right). \quad (3.51)$$

The last step in (3.50) is found with

$$\mathcal{L}_{\xi}A_{\mu} = 0 \implies \mathcal{L}_{\xi}(F^{\mu\alpha}A_{\alpha}) = 0. \quad (3.52)$$

Assuming that the gauge field falls off fast enough at the spatial infinity, one has

$$\int_V (d^{n-1}x)_{\mu} \nabla_{\nu} \tau^{\mu\nu} = - \oint_H (d^{n-2}x)_{\mu\nu} \tau^{\mu\nu}. \quad (3.53)$$

For the first term in $\tau^{\mu\nu}$,

$$\begin{aligned} & - \oint_H (d^{n-2}x)_{\mu\nu} \left(-\frac{1}{2}F^{\mu\nu}A_{\alpha}\xi^{\alpha} \right) = \frac{1}{2}\Phi_H \oint_H (d^{n-2}x)_{\mu\nu} F^{\mu\nu} \\ &= \frac{1}{2}\Phi_H \left[\oint_{+\infty} (d^{n-2}x)_{\mu\nu} F^{\mu\nu} - \int_V (d^{n-1}x)_{\mu} \nabla_{\nu} F^{\mu\nu} \right] \\ &= \frac{1}{2}\Phi_H \oint_{+\infty} (d^{n-2}x)_{\mu\nu} F^{\mu\nu} = 8\pi\Phi_H Q, \end{aligned} \quad (3.54)$$

where

$$\Phi_H = (A^\alpha \xi_\alpha)|_H, \quad Q = \frac{1}{16\pi} \oint_{+\infty} (d^{n-2}x)_{\mu\nu} F^{\mu\nu} = \frac{1}{16\pi} \oint_{+\infty} *dA. \quad (3.55)$$

In deriving (3.54), we have used (3.49) and that fact that Φ_H is a constant on the horizon. For the second term in $\tau^{\mu\nu}$,

$$\begin{aligned} & \oint_H (d^{n-2}x)_{\mu\nu} A_\alpha \left(F^{\mu\alpha} \xi^\nu - F^{\nu\alpha} \xi^\mu \right) \\ &= \oint_H dA \xi_\mu n_\nu A_\alpha \left(F^{\mu\alpha} \xi^\nu - F^{\nu\alpha} \xi^\mu \right) = \oint_H dA \xi_\mu A_\alpha F^{\mu\alpha} \\ &= - \oint_H dA A_\mu \nabla^\mu (A_\nu \xi^\nu) = - \oint_H dA g^{\mu\nu} A_\mu \partial_\nu (A_\rho \xi^\rho) \\ &= 0. \end{aligned} \quad (3.56)$$

Here we have used $(n_\nu \xi^\nu)|_H = 1$ and $(\xi_\nu \xi^\nu)|_H = 0$. The last step is obtained with the help of assumptions made around (3.118) and (3.119). For most of the commonly known solutions, one has both the metric and the matter fields depending only on r and θ^i , while $\xi^r = \xi^i = A_r = A_i = 0$. As a result, one needs $g^{(r,i)(t,a)} \neq 0$ to get a non-vanishing result from (3.56). However, $g^{(r,i)(t,a)} = 0$ is exactly what one has for (3.118) and (3.119). Combining (3.50), (3.54) and (3.56), we have

$$\frac{n-3}{n-2} M = \Omega_a J_a + \Phi_H Q + \frac{\kappa A}{8\pi}. \quad (3.57)$$

The variational form is given by

$$\delta M = \Omega_a \delta J_a + \Phi_H \delta Q + \frac{\kappa \delta A}{8\pi}. \quad (3.58)$$

One can identify Φ_H as the potential at the horizon (we assume the potential is zero at the spatial infinity) and Q as the total charge.

b. Other Methods

One would have expected the above Komar integral construction to work also for solutions in the Anti-de Sitter (AdS) background. However, the process is jeopardized by the divergence showing up in the integral for the energy [119]. As a result, several alternatives to define the energy have been developed over the years (for refs, see, e.g. [122]). For these alternatives, one of the biggest concerns (apart from the usual requirement that the charge should behave like an energy) is that it should satisfy the first law of thermodynamics. Since the first law does not play a role in most of the alternative definitions of the energy, one basically has to check if it is satisfied case by case. Because of this problem, it is also natural to try to use the first law itself to define the energy, in cases when all other quantities are known (see, e.g. [128]).

In this subsection, we shall briefly discuss two interesting methods to define the energy in AdS background. Both methods have their merits and limitations. The first is the one developed by Abbott and Deser [123]. Their method is attractive in that the idea is very simple. In this method, one starts with Einstein's field equations with a negative cosmological constant $\Lambda < 0$,

$$E_{\mu\nu}^{\Lambda} = 8\pi G_{(n)} T_{\mu\nu}, \quad E_{\mu\nu}^{\Lambda} = R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \Lambda g_{\mu\nu}. \quad (3.59)$$

Given a solution, one perturbs the metric around that of a chosen background $\bar{g}_{\mu\nu}$, $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. Then one linearize $E_{\mu\nu}^{\Lambda}$ to define an effective stress energy tensor,

$$\mathcal{T}_{\mu\nu} = \frac{1}{8\pi G_{(n)}} E_{\mu\nu}^{\Lambda} \Big|_{\text{linearized on } h_{\mu\nu}}. \quad (3.60)$$

One can check that $\bar{\nabla}_{\mu} \mathcal{T}^{\mu\nu} = 0$. Then with the help of a Killing vector $\bar{\xi}$ on the background $\bar{\nabla}_{\mu} \bar{\xi}_{\nu} + \bar{\nabla}_{\nu} \bar{\xi}_{\mu} = 0$, one can define a conserved current $\mathcal{J}^{\mu} = \mathcal{T}^{\mu\nu} \bar{\xi}_{\nu}$. A

conserved charge can then be defined by

$$Q = \int_V d^{n-1}x \sqrt{-\bar{g}} \mathcal{T}^{t\mu} \bar{\xi}_\mu = \oint_{\partial V} d^{n-2}S_i \sqrt{-\bar{g}} \mathcal{F}^{ti}, \quad (3.61)$$

with

$$\begin{aligned} \mathcal{F}^{\mu\rho} &= \frac{1}{16\pi G_{(n)}} [\bar{\xi}_\nu (\bar{\nabla}^\mu h^{\rho\nu} - \bar{\nabla}^\rho h^{\mu\nu}) + \bar{\xi}^\mu \bar{\nabla}^\rho h - \bar{\xi}^\rho \bar{\nabla}^\mu h + h \bar{\nabla}^\mu \bar{\xi}^\rho \\ &\quad - \bar{\xi}^\mu \bar{\nabla}_\nu h^{\rho\nu} + \bar{\xi}^\rho \bar{\nabla}_\nu h^{\mu\nu} + h^{\mu\nu} \bar{\nabla}^\rho \bar{\xi}_\nu - h^{\rho\nu} \bar{\nabla}^\mu \bar{\xi}_\nu] \\ &= \frac{1}{16\pi G_{(n)}} [\bar{\xi}_\nu \bar{\nabla}_\alpha K^{\mu\rho\nu\alpha} - K^{\mu\alpha\nu\rho} \bar{\nabla}_\alpha \bar{\xi}_\nu], \end{aligned} \quad (3.62)$$

and

$$\begin{aligned} K^{\mu\nu\rho\sigma} &= \bar{g}^{\mu\sigma} H^{\nu\rho} + \bar{g}^{\nu\rho} H^{\mu\sigma} - \bar{g}^{\mu\rho} H^{\nu\sigma} - \bar{g}^{\nu\sigma} H^{\mu\rho}, \\ H^{\mu\nu} &= h^{\mu\nu} - \frac{1}{2} \bar{g}^{\mu\nu} h, \quad h = g^{\mu\nu} h_{\mu\nu}. \end{aligned} \quad (3.63)$$

The problem with the Abbott-Deser method is that there is ambiguity related to the choice of the metric for the background AdS spacetime, which can lead to results upsetting the first law of thermodynamics [124].

The second method is the one developed by Ashtekar, Magnon and Das (AMD) [120, 121]. The charge is given by

$$Q_\xi = -\frac{\ell}{8\pi G_{(n)}(n-3)} \oint_{\partial V} \mathcal{E}_{ab} \xi^a dS^b, \quad (3.64)$$

where \mathcal{E}_{ab} is defined in [121], and is proportional to the Weyl tensor. This way of defining the charge is superior than the Abbott-Deser method in that there is no ambiguity in the definition of the charge, and the resulted energy is consistent with the first law of the thermodynamics for black holes [124]. The only problem is that (3.64) is based on the Weyl tensor, and so it is not applicable to dimensions lower than four, in which cases the Weyl tensor vanishes identically.

3. The Asymptotic Symmetries

Asymptotic symmetries are transformations that leave the metric invariant up to what is allowed by given boundary conditions. One convenient way to treat asymptotic symmetries is the covariant phase space method as in [117, 130], which is also good for exact symmetries. The formalism was first used to calculate the central charge of conformal symmetries related to a black hole horizon in [56]. After that, there have been a lot of further developments. Some examples can be found in [57, 58, 126, 127].

To motivate for the covariant phase space method, one starts with the classical mechanics.⁴ The Lagrangian is given by $L = L(q, \dot{q})$, where $q = q(t)$ describes the classical trajectory of a particle. For a small variation of the path,

$$\delta L = \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right). \quad (3.65)$$

The equation of motion is given by

$$E = \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0. \quad (3.66)$$

When this is linearized, one has

$$\delta E = \frac{\partial^2 L}{\partial q^2} \delta q + \frac{\partial^2 L}{\partial \dot{q} \partial q} \delta \dot{q} - \delta \dot{p} = 0, \quad p = \frac{\partial L}{\partial \dot{q}}. \quad (3.67)$$

From the boundary term in (3.65), one can define $\Theta(q, \delta) = p \delta q$ and

$$\begin{aligned} \Omega(q; \delta_1, \delta_2) &= \delta_1 \Theta(q, \delta_2) - \delta_2 \Theta(q, \delta_1) \\ &= \delta_1 p \delta_2 q - \delta_2 p \delta_1 q, \end{aligned} \quad (3.68)$$

where δ_1 and δ_2 stands for two independent variations. Notice that $\Omega(q; \delta_1, \delta_2)$ is time

⁴This treatment closely follows a talk given by R. M. Wald at the conference: *ADM-50: A celebration of current GR innovation*, Texas A&M University (2009).

independent if both $\delta_1 q$ and $\delta_2 q$ satisfy (3.67),

$$\frac{d\Omega(q; \delta_1, \delta_2)}{dt} = \delta_1 \dot{p} \delta_2 q + \delta_1 p \delta_2 \dot{q} - \delta_2 \dot{p} \delta_1 q - \delta_2 p \delta_1 \dot{q} = 0. \quad (3.69)$$

The Hamiltonian of the system can now be defined as

$$\delta H = \Omega\left(q; \delta, \frac{d}{dt}\right) = \delta \Theta\left(q, \frac{d}{dt}\right) - \frac{d}{dt} \Theta(q, \delta) = \delta p \dot{q} - \dot{p} \delta q. \quad (3.70)$$

Here we have taken the liberty to generalize δ to other possible operators, such as d/dt . In the case of a curved spacetime, one might also use the Lie derivative \mathcal{L}_ξ . It follows that

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}. \quad (3.71)$$

Using generalized coordinates, $\phi^a = \{q, p\}$, $a = 1, 2$, one can write

$$\Omega(\phi^a; \delta_1, \delta_2) = \Omega_{ab} \delta_1 \phi^a \delta_2 \phi^b, \quad (\Omega_{ab}) = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}. \quad (3.72)$$

Let (Ω^{ab}) be the inverse of (Ω_{ab}) ,

$$(\Omega^{ab}) = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad (3.73)$$

the Poisson bracket of any two functions is then given by

$$\{f, g\}_{P.B.} = \Omega^{ab} \partial_a f \partial_b g = \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q}. \quad (3.74)$$

A special example is that, for $f = f(q, p)$,

$$\frac{df}{dt} = \frac{\partial f}{\partial q} \dot{q} + \frac{\partial f}{\partial p} \dot{p} = \frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q} = \{f, H\}_{P.B.}. \quad (3.75)$$

For a more general system, there can be more coordinates than just $\{q, p\}$ and Ω_{ab} can be more complicated than in (3.72). By analogy to (3.70), one can try to

construct a charge Q_ξ corresponding to any symmetric transformation δ_ξ ,

$$\delta Q_\xi = \Omega(\phi^a; \delta, \delta_\xi) = \Omega_{ab} \delta \phi^a \delta_\xi \phi^b. \quad (3.76)$$

To make Q_ξ a physically meaningful charge, the variation (3.76) needs to be integrable and $\Omega(\phi^a; \delta, \delta_\xi)$ needs to be constant in time. This will put extra constraints on $\delta \phi^a$ and $\delta_\xi \phi^a$, just as in the case above. Given two charges as defined in (3.76), the Poisson bracket is

$$\left\{ Q_\xi, Q_\zeta \right\}_{P.B.} = \Omega^{ab} \frac{\delta Q_\xi}{\delta \phi^a} \frac{\delta Q_\zeta}{\delta \phi^b} = \Omega(\phi^a; \delta_\xi, \delta_\zeta). \quad (3.77)$$

This result will play a central role in the treatment that follows.

Now consider a system with the Lagrangian density $\mathcal{L} = \mathcal{L}(\phi^a, \partial_\mu \phi^a, \partial_\mu \partial_\nu \phi^a, \dots)$.

The actions is

$$S = \int_{\mathcal{M}} \mathbf{L}, \quad \mathbf{L} = \mathcal{L} \sqrt{|g|} \, d^n x = \mathcal{L} * \mathbf{1}. \quad (3.78)$$

A symmetric transformation should leave the integrand \mathbf{L} invariant or up to a total derivative which integrates to zero,

$$\delta_\epsilon \mathbf{L} = d\mathbf{M}_\epsilon, \quad \delta S = \int_{\mathcal{M}} d\mathbf{M}_\epsilon = \oint_{\partial \mathcal{M}} \mathbf{M}_\epsilon = 0. \quad (3.79)$$

On the other hand,

$$\delta_\epsilon \mathbf{L} = E_a \delta_\epsilon \phi^a * \mathbf{1} + d\Theta(\phi^a, \delta_\epsilon), \quad (3.80)$$

where all the terms involving a derivative on $\delta_\epsilon \phi^a$ have been moved into the $d\Theta$ term.

It is easy to see that $E_a = 0$ is the usual Euler-Lagrange equation for ϕ^a . From (3.79) and (3.80), one can define a Noether current,

$$\mathbf{J}_\epsilon = \Theta(\phi^a, \delta_\epsilon) - \mathbf{M}_\epsilon, \quad (3.81)$$

which becomes a closed form when the equations of motion are satisfied, $d\mathbf{J}_\epsilon = -E_a \cdot \delta_\epsilon \phi^a * \mathbf{1}$. So when $E_a = 0$, one should locally have $\mathbf{J}_\epsilon = d\mathbf{Q}_\epsilon$, with \mathbf{Q}_ϵ being

some $n - 2$ form. Now with appropriate boundary conditions, a conserved charge can be defined as

$$Q_\epsilon = \int_V d\mathbf{Q}_\epsilon = \oint_{\partial V} \mathbf{Q}_\epsilon, \quad (3.82)$$

where V is a space-like slice of the spacetime manifold \mathcal{M} . The charge \mathbf{Q}_ϵ is defined up to an arbitrary closed form, but this ambiguity drops out in (3.82).

For a transformation generated by the Lie derivative, $\delta_\xi \phi^a = \mathcal{L}_\xi \phi^a$, one has

$$\begin{aligned} \delta_\xi \mathbf{L} &= E_a \cdot \mathcal{L}_\xi \phi^a * \mathbf{1} + d\Theta(\phi^a, \mathcal{L}_\xi) \\ &= \mathcal{L}_\xi \mathbf{L} = d(i_\xi \mathbf{L}). \end{aligned} \quad (3.83)$$

The Noether current (3.81) is

$$\mathbf{J}_\xi = \Theta(\phi^a, \mathcal{L}_\xi) - i_\xi \mathbf{L}. \quad (3.84)$$

By analogy to (3.68), one can define

$$\Omega(\phi^a; \delta_1, \delta_2) = \int_V \mathbf{w}(\phi^a; \delta_1, \delta_2), \quad (3.85)$$

$$\mathbf{w}(\phi^a; \delta_1, \delta_2) = \delta_1 \Theta(\phi^a, \delta_2) - \delta_2 \Theta(\phi^a, \delta_1). \quad (3.86)$$

The quantity $\Omega(\phi^a; \delta_1, \delta_2)$ is conserved if

$$d\mathbf{w}(\phi^a; \delta_1, \delta_2) = 0 \implies \oint_{\partial \mathcal{M}} \mathbf{w} = \int_{\mathcal{M}} d\mathbf{w} = 0. \quad (3.87)$$

Notice that,

$$0 = (\delta_1 \delta_2 - \delta_2 \delta_1)(\mathcal{L} * \mathbf{1}) \iff \delta_1 \delta_2 \phi^a = \delta_2 \delta_1 \phi^a, \quad (3.88)$$

$$= (\delta_1 E_a \delta_2 \phi^a - \delta_2 E_a \delta_1 \phi^a) * \mathbf{1} + d\mathbf{w}(\phi^a; \delta_1, \delta_2). \quad (3.89)$$

As a result,

$$d\mathbf{w}(\phi^a; \delta_1, \delta_2) = 0 \implies \delta_1 E_a = \delta_2 E_a = 0. \quad (3.90)$$

So $\delta_1\phi^a$ and $\delta_2\phi^a$ must both satisfy the linearized equations of motion for ϕ^a , in order that $\Omega(\phi^a; \delta_1, \delta_2)$ can be constant in time. When this condition is satisfied, one can try to construct a charge corresponding to $\delta_\xi = \mathcal{L}_\xi$, by analogy to (3.70),

$$\delta Q_\xi = \Omega(\phi^a; \delta, \mathcal{L}_\xi) = \int_V \mathbf{w}(\phi^a; \delta, \mathcal{L}_\xi). \quad (3.91)$$

The variation of the Noether current (3.84) is

$$\begin{aligned} \delta \mathbf{J}_\xi &= \delta \Theta(\phi^a, \mathcal{L}_\xi) - i_\xi \delta \mathbf{L} \\ &= \delta \Theta(\phi^a, \mathcal{L}_\xi) - \mathcal{L}_\xi \Theta(\phi^a, \delta) + d \left[i_\xi \Theta(\phi^a, \delta) \right], \end{aligned} \quad (3.92)$$

where the second line is obtained for $E_a = 0$. As a result,

$$\begin{aligned} \mathbf{w}(\phi^a; \delta, \mathcal{L}_\xi) &= \delta \Theta(\phi^a, \mathcal{L}_\xi) - \mathcal{L}_\xi \Theta(\phi^a, \delta) = d \mathbf{k}_\xi(\phi^a, \delta), \\ \implies \delta Q_\xi &= \oint_{\partial V} \mathbf{k}_\xi(\phi^a, \delta), \end{aligned} \quad (3.93)$$

with

$$\mathbf{k}_\xi(\phi^a, \delta) = \delta \mathbf{Q}_\xi - i_\xi \Theta(\phi^a, \delta). \quad (3.94)$$

Note that $\delta(\mathcal{L}_\xi \phi^a) = \mathcal{L}_\xi(\delta \phi^a)$, so both δ and \mathcal{L}_ξ satisfy the assumption made about the operators δ_1 and δ_2 in (3.88). From (3.93),

$$Q_\xi(\phi) = \int_{\bar{\phi}}^{\phi} \delta Q_\xi + Q_\xi(\bar{\phi}) = \int_{\bar{\phi}}^{\phi} \oint_{\partial V} \mathbf{k}_\xi(\phi^a, \delta) + Q_\xi(\bar{\phi}), \quad (3.95)$$

where $Q_\xi(\bar{\phi})$ is the value of the charge on a given background. For the charge $Q_\xi(\phi)$ to be well defined, one expects the integral to be finite. Now given two such charges (say Q_ξ and Q_ζ), the Poisson bracket is found by analogy to (3.77),

$$\left\{ Q_\xi, Q_\zeta \right\}_{PB} = \Omega(\phi^a; \mathcal{L}_\xi, \mathcal{L}_\zeta) = \oint_{\partial V} \mathbf{k}_\xi(\phi^a, \mathcal{L}_\zeta). \quad (3.96)$$

It was shown in [131, 132] that with appropriate boundary conditions, the Poisson

bracket $\{Q_\xi, Q_\zeta\}_{PB}$ of any differentiable generators Q_ξ and Q_ζ takes the form

$$\{Q_\xi, Q_\zeta\}_{PB} = Q_{[\xi, \zeta]} + K[\xi, \zeta], \quad (3.97)$$

where $K[\xi, \zeta]$ is a potential central extension to the algebra. It is demonstrated in [132] that a constant shift in the charges will not affect the nontrivial part of $K[\xi, \zeta]$. Using this, we can shift the charges by some constant and let $Q_{[\xi, \zeta]}(\bar{\phi}) = 0$ in a chosen background. Then we get

$$K[\xi, \zeta] = \{Q_\xi, Q_\zeta\}_{PB} = \oint_{\partial V} \mathbf{k}_\xi(\bar{\phi}^a, \mathcal{L}_\zeta). \quad (3.98)$$

Note if instead of using (3.91), had we chosen to define

$$\delta Q_\xi = -\Omega(\phi^a; \delta, \mathcal{L}_\xi) = -\int_V \mathbf{w}(\phi^a; \delta, \mathcal{L}_\xi), \quad (3.99)$$

we would have got

$$K[\xi, \zeta] = \{Q_\xi, Q_\zeta\}_{PB} = -\Omega(\phi^a; \mathcal{L}_\xi, \mathcal{L}_\zeta) = -\oint_{\partial V} \mathbf{k}_\xi(\phi^a, \mathcal{L}_\zeta). \quad (3.100)$$

This result was used in the calculation of the Kerr/CFT correspondence [26].

In the case of pure gravity supplemented with a cosmological constant, the Lagrangian density is given by

$$\mathcal{L} = \frac{R - 2\Lambda}{16\pi}. \quad (3.101)$$

For an infinitesimal variation of the metric,

$$\delta \mathbf{L} = \frac{1}{16\pi} \left(-R^{\mu\nu} + \frac{R - 2\Lambda}{2} g^{\mu\nu} + \nabla^\mu \nabla^\nu - g^{\mu\nu} \nabla_\rho \nabla^\rho \right) \delta g_{\mu\nu} * \mathbf{1}. \quad (3.102)$$

Einstein's equations are

$$E^{\mu\nu} = R^{\mu\nu} - \frac{R - 2\Lambda}{2} g^{\mu\nu} = 0, \quad (3.103)$$

$$\implies R_{\mu\nu} = \frac{2\Lambda}{n-2} g_{\mu\nu}, \quad R = \frac{2n\Lambda}{n-2}. \quad (3.104)$$

When (3.103) is linearized, one has

$$0 = \delta E_{\mu\nu} = \frac{1}{2} \left[\nabla^\rho (\nabla_\mu h_{\nu\rho} + \nabla_\nu h_{\mu\rho}) - \square h_{\mu\nu} - \nabla_\mu \nabla_\nu h \right] - \frac{1}{2} \left[\nabla_\mu \nabla_\nu h^{\mu\nu} - \square h - R^{\rho\sigma} h_{\rho\sigma} \right] g_{\mu\nu} - \frac{R - 2\Lambda}{2} h_{\mu\nu}, \quad (3.105)$$

where $h_{\mu\nu} = \delta g_{\mu\nu}$ and $h = g^{\mu\nu} h_{\mu\nu}$. Taking the trace of (3.105), one has

$$\nabla_\mu \nabla_\nu h^{\mu\nu} - \square h - R^{\mu\nu} h_{\mu\nu} = 0. \quad (3.106)$$

From (3.83),

$$\begin{aligned} \Theta(g_{\mu\nu}, \delta) &= \frac{1}{16\pi} (d^{n-1}x)_\mu \left[\nabla_\nu h^{\mu\nu} - \nabla^\mu h \right], \\ \Rightarrow i_\xi \Theta(g_{\mu\nu}, \delta) &= \frac{1}{16\pi} (d^{n-2}x)_{\mu\nu} 2\xi^\nu (\nabla_\nu h^{\mu\nu} - \nabla^\mu h) \\ &= \frac{1}{16\pi} (d^{n-2}x)_{\mu\nu} (-I_{\Theta_\xi}^{\mu\nu}), \end{aligned} \quad (3.107)$$

where

$$I_{\Theta_\xi}^{\mu\nu} = \xi^\mu \nabla_\rho h^{\nu\rho} - \xi^\nu \nabla_\rho h^{\mu\rho} + \xi^\nu \nabla^\mu h - \xi^\mu \nabla^\nu h. \quad (3.108)$$

The Noether current (3.84) is

$$\begin{aligned} \mathbf{J}_\xi &= \frac{1}{16\pi} (d^{n-1}x)_\mu \left[\nabla^\nu \nabla^\mu \xi_\nu + \square \xi^\mu - 2\nabla^\mu \nabla^\nu \xi_\nu - (R - 2\Lambda) \xi^\mu \right] \\ &= -\frac{1}{16\pi} (d^{n-1}x)_\mu \nabla_\nu \left[\nabla^\mu \xi^\nu - \nabla^\nu \xi^\mu \right], \\ \Rightarrow \mathbf{Q}_\xi &= -\frac{1}{16\pi} (d^{n-2}x)_{\mu\nu} (\nabla^\mu \xi^\nu - \nabla^\nu \xi^\mu), \end{aligned} \quad (3.109)$$

where we have used (3.103). Note that $\delta \mathbf{Q}_\xi = \frac{1}{16\pi} (d^{n-2}x)_{\mu\nu} I_{Q_\xi}^{\mu\nu}$, with

$$\begin{aligned} I_{Q_\xi}^{\mu\nu} &= -\frac{h}{2} (\nabla^\mu \xi^\nu - \nabla^\nu \xi^\mu) + h^{\mu\rho} \nabla_\rho \xi^\nu - h^{\nu\rho} \nabla_\rho \xi^\mu \\ &\quad - (\nabla^\mu h^{\nu\rho} - \nabla^\nu h^{\mu\rho}) \xi_\rho. \end{aligned} \quad (3.110)$$

From (3.94), one gets that

$$\begin{aligned}
\mathbf{k}_\xi(g_{\mu\nu}, \delta) &= \frac{1}{16\pi}(d^{n-2}x)_{\mu\nu}k^{\mu\nu}, \\
k^{\mu\nu} = I_{Q_\xi}^{\mu\nu} + I_{\Theta_\xi}^{\mu\nu} &= \xi^\nu \nabla^\mu h - \xi^\nu \nabla_\rho h^{\mu\rho} + \frac{h}{2} \nabla^\nu \xi^\mu - h^{\nu\rho} \nabla_\rho \xi^\mu + \xi_\rho \nabla^\nu h^{\mu\rho} \\
&\quad - (\mu \leftrightarrow \nu).
\end{aligned} \tag{3.111}$$

This result matches with that given in [133] up to a trivial term. Note [26] uses a formula for $\mathbf{k}_\xi(g_{\mu\nu}, \delta)$ with the opposite sign, for which to make sense, we need to use (3.99) and (3.100).

To finish, notices that one may also try to use the covariant phase method for exact symmetries. In this process, it is necessary to derive \mathbf{Q}_ξ from (3.109) *before* taking ξ to be an exact Killing vector $\mathcal{L}_\xi g_{\mu\nu} = 0$. Otherwise, one would have found $\mathcal{J}_\xi = 0$ and a meaningful result for \mathbf{Q}_ξ cannot be derived. Now when ξ is a Killing vector, one has for $\mathbf{w}(\phi^a; \delta, \mathcal{L}_\xi)$ in (3.93), $\Theta(\phi^a, \mathcal{L}_\xi) = 0$ and

$$\begin{aligned}
\mathcal{L}_\xi \Theta(\phi^a, \delta) &= \frac{1}{16\pi}(d^{n-1}x)_\mu K^\mu, \\
K^\mu &= -\nabla_\nu I_{\Theta_\xi}^{\mu\nu} + \xi^\mu \nabla_\nu (\nabla_\rho h^{\nu\rho} - \nabla^\nu h).
\end{aligned} \tag{3.112}$$

If $h_{\mu\nu} = \delta g_{\mu\nu}$ only involves varying the parameters in the solution, the symmetry properties of the metric will not be affected, and so

$$\begin{aligned}
\mathcal{L}_\xi h_{\mu\nu} &= \xi^\rho \nabla_\rho h_{\mu\nu} + h_{\rho\nu} \nabla_\mu \xi^\rho + h_{\mu\rho} \nabla_\nu \xi^\rho = 0, \\
&\implies \xi^\rho \nabla_\rho h = 0.
\end{aligned} \tag{3.113}$$

Using this condition, one can show that $K^\mu = 0$. As a result, $\mathbf{w}(\phi^a; \delta, \mathcal{L}_\xi) = 0$ and from (3.93),

$$0 = \int_V \mathbf{w}(\phi^a; \delta, \mathcal{L}_\xi) = \oint_{\partial V} [\delta \mathbf{Q}_\xi - i_\xi \Theta(\phi^a, \delta)]$$

$$= \left(\int_{\infty} - \int_H \right) \left[\delta \mathbf{Q}_{\xi} - i_{\xi} \Theta(\phi^a, \delta) \right]. \quad (3.114)$$

For all known stationary and axisymmetric black holes (3.118) and (3.119), the Killing vectors are $\xi = k = \partial_t$ and $\xi = \ell_a = \partial_{\phi^a}$. At the spatial infinity, $r \rightarrow +\infty$, the term $i_{\xi} \Theta$ contributes only when $\xi = k$ and for cases that have been checked, $I_{\Theta_k}^{\mu\nu} = \frac{1}{n-3} I_{Q_k}^{\mu\nu}$. It was shown in [7] (in asymptotically flat spacetimes) that for $\xi = k + \Omega_a \ell_a$,

$$- \int_H \left[\delta \mathbf{Q}_{\xi} - i_{\xi} \Theta(\phi^a, \delta) \right] = \frac{\kappa \delta A}{8\pi}. \quad (3.115)$$

Now if one defines

$$\begin{aligned} \mathcal{E} &= -\frac{n-2}{n-3} \int_{\infty} \mathbf{Q}_k = \frac{n-2}{16\pi(n-3)} \int_{\infty} *dk, \\ \mathcal{J}_a &= \int_{\infty} \mathbf{Q}_{\ell_a} = -\frac{1}{16\pi} \int_{\infty} *d\ell_a, \end{aligned} \quad (3.116)$$

one has from (3.114) that

$$\delta \mathcal{E} = \Omega_a \delta \mathcal{J}_a + \frac{\kappa \delta A}{8\pi}. \quad (3.117)$$

This is just the first law of thermodynamics for black holes derived in (3.45). But here the normalization of the charges follows naturally from the equations involved, rather than the more or less random pick as in (3.43).

B. Entropies via the Kerr/CFT Correspondence

In this section, we calculate the entropy for various black holes by using the Kerr/CFT correspondence [26]. The calculation has been shown to work for all the cases that were checked (for refs., see [63]). On the physics side, the success of the method provides strong support for the long held belief the the entropy of a black hole is accounted for by the dynamics on the horizon [54, 55, 56, 57, 59, 60, 52]. On the pure theory side, the success also demonstrated the power of the proposed gauge/gravity

duality [13, 14, 15, 16]. However, notice that the Kerr/CFT correspondence only works for extremal and near extremal black holes [63, 64]. What's more, although one can use it to calculate the entropy, very little is known about the nature of the quantum states living on the horizon.

1. The Kerr/CFT Correspondence for General Extremal Black Hole Solutions

The basics of the Kerr/CFT correspondence is explained in [26] in much detail. In this subsection, we will show that the calculation is applicable to all known extremal stationary and axisymmetric black holes. The work generalizes the results found in [24, 25], which was done with H. Lü, C. N. Pope [24] and H. Lü, C. N. Pope and J. Vazquez-Poritz [25].

In the process, a result found in [134] is playing a crucial role. In [134], it was shown that a particular form of the near-horizon metrics for some black holes found in [135, 136] could be very important to the success of the Kerr/CFT correspondence. Here we will show that such form of the near-horizon metric is valid for all known extremal stationary and axisymmetric black holes. This will be done with the help of two ansätze that are general enough to cover all such known black holes. Then the regularity of the black hole horizon will put extra constraint on the properties of the metrics. The particular form of the near-horizon metric mentioned above then follows as soon as the extremal limit is taken. After this, we will show explicitly that the obtained near-horizon metric is all that we need to successfully calculate the entropy by using the Kerr/CFT correspondence. Thus we show that the calculation of the Kerr/CFT correspondence can be applied to all known extremal stationary and axisymmetric black holes.

In the next subsection, we will supply the general arguments made here with

explicit examples, mostly based on the work done in [24] and [25]. But we will also include some of the examples studied in [134].

a. Ansätze for All Known Stationary and Axisymmetric Black Holes

Here we shall derive two general ansätze which we believe cover all known stationary and axisymmetric black hole solutions. The derivation will be partially based on our experience with all the solutions that are known.

Lets start with some known features of the metrics of stationary and axisymmetric black holes:

- By using the term “stationary and axisymmetric”, one assumes that (i) a coordinate system exists where some of the coordinates can be identified with the asymptotic time direction t and the azimuthal directions ϕ^a , and (ii) the metric does not depend on t nor ϕ^a .
- Among the rest of the coordinates, one coordinate can be singled out as describing the radial direction r . For all known solutions, the position of the black hole horizon ($r = r_H$) is determined by a single function of r : $\Delta(r_H) = 0$.
- All other coordinates are then related to the latitudinal angles θ^i . For a black hole in n dimensional spacetime, there can be $\lfloor \frac{n-1}{2} \rfloor$ independent rotations. So $a = 1, \dots, \lfloor \frac{n-1}{2} \rfloor$ and $i = 1, \dots, \lfloor \frac{n}{2} \rfloor - 1$.
- For all known solutions, one can always chose the coordinate systems so that the metrics do not have any cross terms involving dr or $d\theta^i$.
- Near the black hole horizon, it can either be a term like $dt + f_a(r, \theta^i)d\phi^a$ or a term like $f_a(r, \theta^i)d\phi^a$ playing the role of time.

Metrics reflecting such features can always be written as

$$ds_n^2 = -\frac{\Delta}{f_t} \left[dt + f_a d\phi^a \right]^2 + \frac{f_r}{\Delta} dr^2 + g_{ij} d\theta^i d\theta^j + d\bar{s}_\phi^2, \quad (3.118)$$

$$ds_n^2 = -\frac{\Delta}{f_t} \left[f_a d\phi^a \right]^2 + \frac{f_r}{\Delta} dr^2 + g_{ij} d\theta^i d\theta^j + d\bar{s}_\phi^2, \quad (3.119)$$

where

$$d\bar{s}_\phi^2 = g_{ab} (d\phi^a - \chi_a dt) (d\phi^b - \chi_b dt) + f_{tt} dt^2. \quad (3.120)$$

Note all the functions depend on r and θ^i 's only, while Δ only depends on r . We have allowed $d\theta^i$'s to mix among themselves in (3.118) and (3.119), so both ansätze can describe possibly slightly more general cases than listed above. Also, we include the $f_{tt} dt^2$ term in (3.120) just to make (3.118) and (3.119) as general as possible. The assumption on this term is that it should not play any significant role when it come close to the horizon. As can be seen below, this means $f_{tt} \sim \Delta^2$ as $r \rightarrow r_H$.

We believe that all known stationary and axisymmetric black holes can either be written in the form of (3.118) or in the form of (3.119).

Some extra constraints can be imposed on the functions in (3.118), (3.119) and (3.120) when one approaches the black hole horizon. These constraints are obtained by noticing that there should not be any genuine singularities related to the horizon, which means both the metric and the matter fields (if there are any) should be regular on the horizon in an appropriately chosen coordinate system. In the following, we will only be concerned with the metric.

To see how the regularity of horizon can help us, note that the first two terms in (3.118) can be written as

$$\frac{\Delta}{f_t} \left(- \left[dt + f_a d\phi^a \right]^2 + \frac{f_t f_r}{\Delta^2} dr^2 \right) = -\frac{\Delta}{f_t} \mathcal{A}^2 + 2\sqrt{f_r/f_t} dr \mathcal{A}, \quad (3.121)$$

where

$$\mathcal{A} = dt + f_a d\phi^a + \frac{\sqrt{f_t f_r}}{\Delta} dr. \quad (3.122)$$

The superficial singularity near the horizon comes solely from $\Delta(r_H) = 0$. To make the metric regular on the horizon, one can try to make \mathcal{A} regular first. This can be achieved if there exist functions $h_v = h_v(r)$, $h_a = h_a(r)$ and $h_{\mathcal{A}} = h_{\mathcal{A}}(r, \theta^i)$ being regular on the horizon and satisfying

$$\sqrt{f_t f_r} = h_v + f_a h_a + h_{\mathcal{A}} \Delta + \mathcal{O}(\Delta^2). \quad (3.123)$$

Because then one can write $\mathcal{A} = dv + f_a d\psi^a + h_{\mathcal{A}} dr + \mathcal{O}(\Delta)$, after using the coordinate transformation

$$dv = dt + \frac{h_v(r)}{\Delta(r)} dr, \quad d\psi^a = d\phi^a + \frac{h_a(r)}{\Delta(r)} dr. \quad (3.124)$$

Now if one replaces (t, ϕ^a) by (v, ψ^a) as defined in (3.124), one has from (3.120)

$$\begin{aligned} d\bar{s}_\phi^2 &= g_{ab} \left(d\psi^a - \chi_a dv - \frac{h_a - \chi_a h_v}{\Delta} dr \right) \left(d\psi^b - \chi_b dv - \frac{h_b - \chi_b h_v}{\Delta} dr \right) \\ &\quad + f_{tt} \left(dv - \frac{h_v}{\Delta} dr \right)^2. \end{aligned} \quad (3.125)$$

To make $d\bar{s}_\phi^2$ regular on the horizon, one must have

$$\chi_a = \frac{h_a + h_\chi^a \Delta}{h_v} + \mathcal{O}(\Delta^2), \quad f_{tt} = h_{tt} \Delta^2 + \mathcal{O}(\Delta^3). \quad (3.126)$$

Again, $h_\chi^a = h_\chi^a(r, \theta^i)$ and $h_{tt} = h_{tt}(r, \theta^i)$ must be regular on the horizon. Using these results and keeping only leading order corrections, one has for (3.118) at $r \rightarrow r_H$,

$$\begin{aligned} ds_n^2 &\approx f_r \left\{ -\Delta \frac{(dt + f_a d\phi^a)^2}{(h_v + f_a h_a + h_{\mathcal{A}} \Delta)^2} + \frac{dr^2}{\Delta} \right\} + g_{ij} d\theta^i d\theta^j + h_{tt} \Delta^2 dt^2 \\ &\quad + g_{ab} \left(d\phi^a - \frac{h_a + h_\chi^a \Delta}{h_v} dt \right) \left(d\phi^b - \frac{h_b + h_\chi^b \Delta}{h_v} dt \right). \end{aligned} \quad (3.127)$$

If one repeats the same process for (3.119), one can find that when $r \rightarrow r_H$,

$$ds_n^2 \approx f_r \left\{ -\Delta \frac{(f_a d\phi^a)^2}{(f_a h_a + h_{\mathcal{A}} \Delta)^2} + \frac{dr^2}{\Delta} \right\} + g_{ij} d\theta^i d\theta^j + h_{tt} \Delta^2 dt^2 \\ + g_{ab} \left(d\phi^a - \frac{h_a + h_{\chi}^a \Delta}{h_v} dt \right) \left(d\phi^b - \frac{h_b + h_{\chi}^b \Delta}{h_v} dt \right). \quad (3.128)$$

For a surprisingly large number of solutions, (3.127) with $h_{\mathcal{A}} = h_{\chi}^a = h_{tt} = 0$ are in fact exact (i.e., not an approximation). We will show this with explicit examples in the next subsection.

Strictly speaking, our derivation of (3.127) and (3.128) is by no means the most general one. The whole process rests upon using the coordinate transformation (3.124) to render both \mathcal{A} and $d\bar{s}_{\phi}^2$ finite on the horizon *separately*. One may as well tries to think of other ways to make the whole metric (3.118) finite on the horizon all together. We have made no effort trying in that direction. But one thing to notice is that (3.127) and (3.128) already seem to be general enough to cover all the stationary and axisymmetric solutions that we know.

To calculate the surface gravity for a black hole, lets choose t and ϕ^a so that the coordinate system is static and both t and ϕ^a are canonically normalized. Then the surface gravity is calculated with the particular Killing vector,

$$\xi = \partial_t + \Omega_a \partial_{\phi^a}. \quad (3.129)$$

Here the constants Ω_a 's are chosen to make ξ null on the horizon. They are interpreted as the angular velocities corresponding to the azimuthal angles ϕ^a . To see how Ω_a 's can be calculated, note that for (3.127),

$$\xi^2 = \frac{-f_r \Delta \cdot (1 + f_a \Omega_a)^2}{(h_v + f_a h_a + h_{\mathcal{A}} \Delta)^2} + g_{ab} \left(\Omega_a - \frac{h_a + h_{\chi}^a \Delta}{h_v} \right) \left(\Omega_b - \frac{h_b + h_{\chi}^b \Delta}{h_v} \right) + h_{tt} \Delta^2, \quad (3.130)$$

and for (3.128),

$$\xi^2 = \frac{-f_r \Delta \cdot (f_a \Omega_a)^2}{(f_a h_a + h_{\mathcal{A}} \Delta)^2} + g_{ab} \left(\Omega_a - \frac{h_a + h_{\chi}^a \Delta}{h_v} \right) \left(\Omega_b - \frac{h_b + h_{\chi}^b \Delta}{h_v} \right) + h_{tt} \Delta^2. \quad (3.131)$$

For both cases, to make ξ vanish on the horizon one must have

$$\Omega_a = \frac{h_a^0}{h_v^0}, \quad h_a^0 = h_a(r_H), \quad h_v^0 = h_v(r_H). \quad (3.132)$$

Including corrections to the leading order, one has

$$\frac{h_a}{h_v} = \Omega_a + \Omega'_a \cdot (r - r_H) + \mathcal{O}(r - r_H)^2, \quad \Omega'_a \equiv \left(\frac{h_a}{h_v} \right)' \Big|_{r=r_H}. \quad (3.133)$$

The surface gravity on the horizon can be calculated using

$$\kappa^2 = \frac{g^{\mu\nu} \partial_\mu \lambda \partial_\nu \lambda}{4\lambda} \Big|_H, \quad \lambda = -\xi^2. \quad (3.134)$$

For non-extremal solutions, $\Delta(r) = \Delta'_0 \cdot (r - r_H) + \mathcal{O}(r - r_H)^2$ with $\Delta'_0 = \Delta'(r_H)$. So to leading order,

$$\lambda = -\xi^2 = \frac{f_r^0}{h_v^{02}} \Delta'_0 \cdot (r - r_H) + \mathcal{O}(r - r_H)^2, \quad (3.135)$$

where $f_r^0 = f_r(r_H, \theta^i)$. The surface gravity (3.134) is then given by

$$\begin{aligned} \kappa^2 &= \frac{g^{rr} \partial_r \lambda \partial_r \lambda + g^{ij} \partial_i \lambda \partial_j \lambda}{4\lambda} \Big|_H \\ &= \frac{g^{rr} \partial_r \lambda \partial_r \lambda}{4\lambda} \Big|_H = \frac{\Delta_0'^2}{4h_v^{02}}. \end{aligned} \quad (3.136)$$

So the temperature of the black hole is given by

$$T_H = \frac{\kappa}{2\pi} = \frac{\Delta_0'}{4\pi h_v^0}. \quad (3.137)$$

For an extremal solution, $\Delta = \frac{1}{2} \Delta_0'' \cdot (r - r_H)^2 + \mathcal{O}(r - r_H)^3$ with $\Delta_0'' = \Delta''(r_H)$. So

to the leading order,

$$\lambda = -\xi^2 = \left(\frac{f_r^0 \Delta_0''}{2h_v^0} - g_{ab}^0 \Omega'_a \Omega'_b \right) (r - r_H)^2 + \mathcal{O}(r - r_H)^3, \quad (3.138)$$

where $g_{ab}^0 = g_{ab}(r_H, \theta^i)$. Note Δ_0'' , h_v^0 and Ω'_a are all constants. As a result, the surface gravity (3.134) is

$$\kappa^2 = \frac{g^{rr} \partial_r \lambda \partial_r \lambda + g^{ij} \partial_i \lambda \partial_j \lambda}{4\lambda} \Big|_H = 0 \quad \implies \quad T_H = 0. \quad (3.139)$$

The vanishing of the temperature can also be derived by starting from (3.137), and then take the extremal limit

$$\Delta'_0 \rightarrow 0 \quad \implies \quad T_H \rightarrow 0. \quad (3.140)$$

All these results are valid for both (3.127) and (3.128).

b. The Near-Horizon Metric for Extremal Black Holes

To get the near-horizon metric for an extremal black hole, one can let

$$r \rightarrow r_H + y \lambda r_H, \quad t \rightarrow \frac{2h_v^0}{\lambda r_H \Delta_0''} \tilde{t}, \quad \phi^a \rightarrow \phi^a + \Omega_a \frac{2h_v^0}{\lambda r_H \Delta_0''} \tilde{t}. \quad (3.141)$$

Using $\Delta = \frac{1}{2} \Delta_0'' \cdot (r - r_H)^2 + \mathcal{O}(r - r_H)^3$ and after sending $\lambda \rightarrow 0$, one has for both (3.127) and (3.128),

$$\begin{aligned} ds^2 = & \frac{2f_r^0}{\Delta_0''} \left(-y^2 d\tilde{t}^2 + \frac{dy^2}{y^2} \right) + g_{ij}^0 d\theta^i d\theta^j \\ & + g_{ab}^0 (d\phi^a + k^a y d\tilde{t})(d\phi^b + k^b y d\tilde{t}), \end{aligned} \quad (3.142)$$

where $g_{ij}^0 = g_{ij}(r_H, \theta^i)$, and we have used (3.133) and have defined

$$k^a = -\frac{2h_v^0 \Omega'_a}{\Delta_0''}. \quad (3.143)$$

To get to the global coordinates, let

$$y = r + \sqrt{1+r^2} \cos t, \quad \tilde{t} = \frac{\sqrt{1+r^2} \sin t}{y}. \quad (3.144)$$

Then

$$\begin{aligned} -y^2 d\tilde{t}^2 + \frac{dy^2}{y^2} &= -(1+r^2)dt^2 + \frac{dr^2}{1+r^2}, \\ y d\tilde{t} &= r dt + d \ln \left(\frac{1 + \sqrt{1+r^2} \sin t}{\cos t + r \sin t} \right). \end{aligned} \quad (3.145)$$

So by letting

$$\phi^a \rightarrow \phi^a - k^a \ln \left(\frac{1 + \sqrt{1+r^2} \sin t}{\cos t + r \sin t} \right), \quad (3.146)$$

one can rewrite the near-horizon metric (3.142) as

$$\begin{aligned} ds^2 &= \frac{2f_r^0}{\Delta_0''} \left[-(1+r^2)dt^2 + \frac{dr^2}{1+r^2} \right] + g_{ij}^0 d\theta^i d\theta^j \\ &\quad + g_{ab}^0 (d\phi^a + k^a r dt)(d\phi^b + k^b r dt). \end{aligned} \quad (3.147)$$

The significance of this form of the near-horizon metric was first discovered in [134] and was further demonstrated in [25].

c. The Central Charge for the Dual CFT

Similar to [26], one can try to study the entropy by defining degenerate field configurations, i.e., states whose near-horizon limit is equivalent to (3.147) up to appropriately defined boundary conditions. The symmetry of the corresponding phase space is generated by $[\frac{d-1}{2}]$ commuting generators, namely

$$\xi_m^a = -e^{-im\phi^a} \partial_{\phi^a} - im r e^{-im\phi^a} \partial_r, \quad a = 1, \dots, [\frac{d-1}{2}], \quad (3.148)$$

where d is the dimension of the spacetime. It is then easy to check that

$$i[\xi_m^a, \xi_n^a] = (m - n)\xi_{m+n}^a. \quad (3.149)$$

These transformations generate $[\frac{d-1}{2}]$ commuting Virasoro algebras. For each Virasoro algebra, the phase space can then be identified with that of a two-dimensional conformal field theory. In the case when the corresponding charges $Q_{\xi_m^a}$ in the phase space are well defined, the quantum version of the charges are given by

$$Q_{\xi_m^a} = L_m^a - \alpha\delta_m, \quad (3.150)$$

with α being some irrelevant constant. From (3.95) and (3.111), it is easy to see that if ξ_m^a is scaled by a factor, the right hand side of (3.150) also needs to be scaled by the same factor. Especially, one has

$$Q_{[\xi_m^a, \xi_n^a]} = Q_{-i(m-n)\xi_{m+n}^a} = -i(m-n)(L_{m+n}^a - \alpha\delta_{m+n}). \quad (3.151)$$

So from (3.97),

$$\begin{aligned} [L_m^a, L_n^a] &= i\{Q_{\xi_m^a}, Q_{\xi_n^a}\}_{PB} = i(Q_{[\xi_m^a, \xi_n^a]} + K[\xi_m^a, \xi_n^a]) \\ &= (m-n)L_{m+n} - 2m\alpha\delta_{m+n} + iK[\xi_m^a, \xi_n^a]. \end{aligned} \quad (3.152)$$

Compare this with the usual relation,

$$[L_m^a, L_n^a] = (m-n)L_{m+n}^a + \frac{c^a}{12}m(m^2-1)\delta_{m+n}, \quad (3.153)$$

one gets

$$K[\xi_m^a, \xi_n^a] = -i\frac{c^a}{12}m\left(m^2-1 + \frac{24\alpha}{c^a}\right)\delta_{m+n}. \quad (3.154)$$

So the central charge c^a is determined by the coefficient of the m^3 term in $K[\xi_m^a, \xi_n^a]$.

The term linear in m is not so important because α is a free parameter.

The central term $K[\xi_m^a, \xi_n^a]$ corresponding to the near-horizon metric (3.147) is calculated in (B.12),

$$K[\xi_m^a, \xi_n^a] = -\frac{i(m-n)n^2 k^a}{16\pi} \delta_{m+n} \mathcal{A}, \quad (3.155)$$

with \mathcal{A} being the area of the horizon. Comparing this result with (3.154), one has

$$c^a = \frac{3k^a}{2\pi} \mathcal{A}. \quad (3.156)$$

Note this central charge is calculated only with contribution from the Einstein-Hilbert action. For a discussion of contributions from matter fields and from more complicated gravitational theories, see [137, 138, 139].

d. The Entropy

In the following, we shall try to relate the central charge to the entropy by using Cardy's formula. Again following [26], one can adopt the Frolov-Thorne vacuum [140] to provide a definition of the vacuum state for the extremal metric.

Quantum fields for the general (non-extremal) metric (3.118) and (3.119) can be expanded in eigenstates with asymptotic energy ω and angular momentum m_a , with \hat{t} and $\hat{\phi}^a$ dependence $e^{-i\omega\hat{t}+im_a\hat{\phi}^a}$. In terms of the redefined t and ϕ^a coordinates of the extremal near-horizon limit, given by (3.141), we have

$$e^{-i\omega\hat{t}+im_a\hat{\phi}^a} = e^{-in_R t + in_L^a \phi^a}, \quad (3.157)$$

with⁵

$$n_L^a = m_a, \quad n_R = \frac{2\tilde{h}_v^0}{\tilde{\Delta}_0'' r_H \lambda} (w - m_a \tilde{\Omega}_a). \quad (3.158)$$

⁵For the rest of the *paragraph*, any quantities from the extremal solution are distinguished with a tilde.

The left-moving and right-moving temperatures T_L and T_R are then defined by writing the Boltzmann factor as

$$e^{-(\omega - m_a \Omega_a)/T_H} = e^{-n_L^a/T_L^a - n_R/T_R}. \quad (3.159)$$

As a result,

$$T_L^a = \frac{T_H}{\tilde{\Omega}_a - \Omega_a}, \quad T_R = \frac{2\tilde{h}_v^0}{\tilde{\Delta}_0'' r_H \lambda} T_H. \quad (3.160)$$

For a solution where the parameter corresponding to the rotation Ω_a is given by ℓ_a , the extremal limit for the temperatures are obtained by taking ℓ_a to its extremal value $\tilde{\ell}_a$. On the horizon,

$$\Delta(r_H) = 0 \implies 0 = \frac{d\Delta(r_H)}{d\ell_a} = \frac{\partial\Delta(r_H)}{\partial\ell_a} + \frac{\partial\Delta(r_H)}{\partial r_H} \frac{dr_H}{d\ell_a}. \quad (3.161)$$

Because $\partial\Delta(r_H)/\partial\ell_a$ must be finite⁶, one has in the extremal limit

$$\frac{\partial\Delta(r_H)}{\partial r_H} \longrightarrow 0 \implies \frac{dr_H}{d\ell_a} = -\frac{\partial\Delta(r_H)}{\partial\ell_a} / \frac{\partial\Delta(r_H)}{\partial r_H} \longrightarrow \infty. \quad (3.162)$$

So in the extremal limit, $T_R = 0$ and

$$\begin{aligned} T_L^a &= \frac{T_H}{\tilde{\Omega}_a - \Omega_a} \Big|_{\ell_a \rightarrow \tilde{\ell}_a} = -\left(\frac{dT_H}{d\ell_a} / \frac{d\Omega_a}{d\ell_a} \right) \Big|_{\ell_a \rightarrow \tilde{\ell}_a} \\ &= -\left(\frac{\partial T_H}{\partial \ell_a} + \frac{\partial T_H}{\partial r_H} \frac{dr_H}{d\ell_a} \right) / \left(\frac{\partial \Omega_a}{\partial \ell_a} + \frac{\partial \Omega_a}{\partial r_H} \frac{dr_H}{d\ell_a} \right) \Big|_{\ell_a \rightarrow \tilde{\ell}_a} \\ &= -\left(\frac{\partial T_H}{\partial r_H} / \frac{\partial \Omega_a}{\partial r_H} \right) \Big|_{\ell_a \rightarrow \tilde{\ell}_a} = -\frac{\tilde{T}_H'(r_H)}{\tilde{\Omega}_a'} = -\frac{\tilde{\Delta}_0''}{4\pi \tilde{\Omega}_a' \tilde{h}_v^0} \\ &= \frac{1}{2\pi k^a}, \end{aligned} \quad (3.163)$$

where we have used (3.143). The result (3.163) was first speculated to be true for general extremal black holes in four dimensions in [137]. It was then generalized to

⁶Note $\partial\Delta(r_H)/\partial\ell_a = 0$ means that the function $\Delta(r)$ does not contain the parameter ℓ_a , which in turn means that r_H is independent of ℓ_a . This is unlikely to happen.

solutions in arbitrary dimensions in [134] based on all the examples that are studied. Here we have shown that (3.163) is true for all known extremal stationary and axisymmetric black holes.

From Cardy's formula for the entropy of a unitary conformal field theory at temperature T_L , the microscopic entropy is given by (no summation over a)

$$\begin{aligned} S &= \frac{1}{3} \pi^2 c_L^a T_L^a \\ &= \frac{\mathcal{A}_{rea}}{4}. \end{aligned} \tag{3.164}$$

Here for the second step we have used (3.156) and (3.163), and have identified c_L^a with c^a . We see that this result exactly matches with the Bekenstein-Hawking entropy. Remembering that the central charge (3.156) only contains contribution from the Einstein-Hilbert action, our result suggests that the matter contribution to the central charge is zero [133, 141].

2. Examples

In this subsection, we will briefly discuss some of the work done in [24] and [25]. But different from the original papers, here we will focus on demonstrating the general applicability of (3.127) and (3.128), since everything else of the Kerr/CFT calculation follows in a straightforward manner. For the same reason, we will also discuss some of the solutions studied in [134].

a. Kerr-NUT-AdS Solutions in Diverse Dimensions

After [26], the conjectured correspondence was then applied to various Kerr-AdS solutions in diverse dimensions [24]. The work was done with Hong Lü and Chris Pope.

The first example is the Kerr-AdS solution in four dimensions. The metric is given by [83]

$$\begin{aligned}
ds^2 &= \rho^2 \left(\frac{d\hat{r}^2}{\Delta} + \frac{d\theta^2}{\Delta_\theta} \right) + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left(ad\hat{t} - \frac{\hat{r}^2 + a^2}{\Xi} d\hat{\phi} \right)^2 - \frac{\Delta}{\rho^2} \left(d\hat{t} - \frac{a \sin^2 \theta}{\Xi} d\hat{\phi} \right)^2, \\
\rho^2 &= \hat{r}^2 + a^2 \cos^2 \theta, \quad \Delta = (\hat{r}^2 + a^2)(1 + \hat{r}^2 \ell^{-2}) - 2M\hat{r}, \\
\Delta_\theta &= 1 - a^2 \ell^{-2} \cos^2 \theta, \quad \Xi = 1 - a^2 \ell^{-2},
\end{aligned} \tag{3.165}$$

which is a solution of the equations $R_{\mu\nu} = -3\ell^{-2} g_{\mu\nu}$. Comparing with (3.121) and (3.122), it is easy to see that

$$\begin{aligned}
\mathcal{A} &= d\hat{t} - \frac{a \sin^2 \theta}{\Xi} d\hat{\phi} + \frac{\rho^2}{\Delta} dr \\
&= d\hat{t} - \frac{a \sin^2 \theta}{\Xi} d\hat{\phi} + \frac{r^2 + a^2 - a^2 \sin^2 \theta}{\Delta} dr, \\
\implies h_v &= r^2 + a^2, \quad h_\phi = a\Xi, \quad h_{\mathcal{A}} = 0.
\end{aligned} \tag{3.166}$$

One sees that the metric is exactly of the form (3.127) with $h_{\mathcal{A}} = h_\chi^\phi = h_{tt} = 0$.

The second example is the five-dimensional rotating black hole with S^3 horizon topology. The solutions was obtained by Hawking, Hunter and Taylor-Robinson [87], satisfying the Einstein equation $R_{\mu\nu} = -4\ell^{-2} g_{\mu\nu}$. This metric, which generalizes the Ricci-flat rotating black hole of Myers and Perry [71], is given by

$$\begin{aligned}
ds^2 &= -\frac{\Delta}{\rho^2} \left(d\hat{t} - \frac{a \sin^2 \theta}{\Xi_a} d\phi_1 - \frac{b \cos^2 \theta}{\Xi_b} d\phi_2 \right)^2 + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left(ad\hat{t} - \frac{(\hat{r}^2 + a^2)}{\Xi_a} d\phi_1 \right)^2 \\
&\quad + \frac{\Delta_\theta \cos^2 \theta}{\rho^2} \left(bd\hat{t} - \frac{(\hat{r}^2 + b^2)}{\Xi_b} d\phi_2 \right)^2 + \frac{\rho^2}{\Delta} d\hat{r}^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 \\
&\quad + \frac{1 + \hat{r}^2 \ell^{-2}}{\hat{r}^2 \rho^2} \left(abd\hat{t} - \frac{b(\hat{r}^2 + a^2) \sin^2 \theta}{\Xi_a} d\phi_1 - \frac{a(\hat{r}^2 + b^2) \cos^2 \theta}{\Xi_b} d\phi_2 \right)^2,
\end{aligned} \tag{3.167}$$

where

$$\begin{aligned}
\Delta &= \frac{1}{\hat{r}^2} (\hat{r}^2 + a^2)(\hat{r}^2 + b^2)(1 + \hat{r}^2 \ell^{-2}) - 2M, \quad \Delta_\theta = 1 - a^2 \ell^{-2} \cos^2 \theta - b^2 \ell^{-2} \sin^2 \theta, \\
\rho^2 &= \hat{r}^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta, \quad \Xi_a = 1 - a^2 \ell^{-2}, \quad \Xi_b = 1 - b^2 \ell^{-2}.
\end{aligned} \tag{3.168}$$

Note that in this coordinate system, the metric is asymptotic to AdS_5 in a rotating frame, with angular velocities $\Omega_{\phi_1}^\infty = -a\ell^{-2}$ and $\Omega_{\phi_2}^\infty = -b\ell^{-2}$. By letting

$$\phi_1 \rightarrow \phi_1 - a\ell^{-2}\hat{t}, \quad \phi_2 \rightarrow \phi_2 - b\ell^{-2}\hat{t}, \quad (3.169)$$

one can change to an asymptotically static coordinate system. The metric is now given by

$$\begin{aligned} ds^2 = & -\frac{\Delta}{\rho^2} \left[\left(1 + \frac{a^2\ell^{-2}\sin^2\theta}{\Xi_a} + \frac{b^2\ell^{-2}\cos^2\theta}{\Xi_b} \right) d\hat{t} - \frac{a\sin^2\theta}{\Xi_a} d\phi_1 - \frac{b\cos^2\theta}{\Xi_b} d\phi_2 \right]^2 \\ & + \frac{\rho^2}{\Delta} d\hat{r}^2 + \frac{\Delta_\theta \sin^2\theta (\hat{r}^2 + a^2)^2}{\rho^2 \Xi_a^2} \left(d\phi_1 - \frac{a(1 + \hat{r}^2\ell^{-2})}{\hat{r}^2 + a^2} d\hat{t} \right)^2 \\ & + \frac{\rho^2}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta \cos^2\theta (\hat{r}^2 + b^2)^2}{\rho^2 \Xi_b^2} \left(d\phi_2 - \frac{b(1 + \hat{r}^2\ell^{-2})}{\hat{r}^2 + b^2} d\hat{t} \right)^2 \\ & + \frac{a^2b^2(1 + \hat{r}^2\ell^{-2})}{\hat{r}^2\rho^2} \left\{ \frac{(\hat{r}^2 + a^2)\sin^2\theta}{a\Xi_a} \left(d\phi_1 - \frac{a(1 + \hat{r}^2\ell^{-2})}{\hat{r}^2 + a^2} d\hat{t} \right) \right. \\ & \quad \left. + \frac{(\hat{r}^2 + b^2)\cos^2\theta}{b\Xi_b} \left(d\phi_2 - \frac{b(1 + \hat{r}^2\ell^{-2})}{\hat{r}^2 + b^2} d\hat{t} \right) \right\}^2. \end{aligned} \quad (3.170)$$

From (3.121) and (3.122),

$$\begin{aligned} \mathcal{A} = & \left(1 + \frac{a^2\ell^{-2}\sin^2\theta}{\Xi_a} + \frac{b^2\ell^{-2}\cos^2\theta}{\Xi_b} \right) d\hat{t} \\ & - \frac{a\sin^2\theta}{\Xi_a} d\phi_1 - \frac{b\cos^2\theta}{\Xi_b} d\phi_2 + \frac{\rho^2}{\Delta} dr. \end{aligned} \quad (3.171)$$

Comparing (3.170) with (3.127), and noticing that \mathcal{A} should be regular on the horizon, we find

$$\begin{aligned} h_v &= \frac{(\hat{r}^2 + a^2)(\hat{r}^2 + b^2)}{\hat{r}^2}, \quad h_1 = \frac{a(1 + \hat{r}^2\ell^{-2})}{\hat{r}^2 + a^2} h_v, \\ h_2 &= \frac{b(1 + \hat{r}^2\ell^{-2})}{\hat{r}^2 + b^2} h_v, \quad h_{\mathcal{A}} = 0. \end{aligned} \quad (3.172)$$

It is easy to see that (3.170) is of the form (3.127) with $h_{\mathcal{A}} = h_\chi^a = h_{tt} = 0$.

In the following, we shall consider the general Kerr-NUT-AdS solutions found in [88], which solve the Einstein equation $R_{\mu\nu} = -(d-1)\ell^{-2}g_{\mu\nu}$. The case of Kerr-AdS

solutions have been studied in [24] and [134]. Since the NUT parameters will not affect anything in the process, here we will include them as well. Also, we will choose to write the metric by analogy to (40) and (48) in [88], which specialize to seven and six dimensions respectively. In even dimensions, $d = 2n$, the metric is given by

$$ds_{2n}^2 = \sum_{i=1}^n \left(\frac{f_i dx_i^2}{X_i} + \frac{X_i}{f_i} \mathcal{A}_i^2 \right), \quad f_i = \prod_{j \neq i} (x_i^2 - x_j^2), \quad (3.173)$$

$$\begin{aligned} \mathcal{A}_i &= dt + \sum_{j \neq i} x_j^2 d\phi_1 + \sum_{j,k \neq i} x_j^2 x_k^2 d\phi_2 + \cdots + \prod_{j \neq i} x_j^2 d\phi_{n-1}, \\ X_i &= 2M_i x_i + \sum_{j=0}^{n-1} c_{2j} x_i^{2j} + g^2 x_i^{2n}. \end{aligned} \quad (3.174)$$

In odd dimensions, $d = 2n + 1$, the metric is given by

$$ds_{2n+1}^2 = ds_{2n}^2 + \frac{c_n}{\prod_{i=1}^n x_i^2} \mathcal{A}_n^2, \quad (3.175)$$

with

$$\begin{aligned} \mathcal{A}_n &= dt + \sum_{i=1}^n x_i^2 d\phi_1 + \sum_{i,j=1}^n x_i^2 x_j^2 d\phi_2 + \cdots + \prod_{i=1}^n x_i^2 d\phi_n, \\ X_i &= (-1)^{\frac{d-1}{2}} \frac{c_n}{x_i^2} + 2M_i + \sum_{j=1}^{n-1} c_{2j} x_i^{2j} + g^2 x_i^{2n}, \\ \mathcal{A}_{i \neq 1} &= dt + \sum_{j \neq 1, i} x_j^2 d\phi_1 + \sum_{j,k \neq 1, i} x_j^2 x_k^2 d\phi_2 + \cdots + \prod_{j \neq 1, i} x_j^2 d\phi_{n-2} \\ &\quad - r^2 \left(d\phi_1 + \sum_{j \neq 1, i} x_j^2 d\phi_2 + \cdots + \prod_{j \neq 1, i} x_j^2 d\phi_{n-1} \right) \\ &= dt - r^2 d\phi_1 + \sum_{j \neq 1, i} x_j^2 (d\phi_1 - r^2 d\phi_2) + \cdots \\ &\quad + \prod_{j \neq 1, i} x_j^2 (d\phi_{n-2} - r^2 d\phi_{n-1}). \end{aligned} \quad (3.176)$$

Note we have wick rotated the radial direction $r^2 \rightarrow -x_1^2$ so that the metrics (3.173) and (3.175) can be put into a compact form. To get back to the Lorentzian signature

black hole metric, one needs to wick rotate back, $x_1^2 \rightarrow -r^2$. Especially, one has

$$\begin{aligned} f_1 &= (-1)^{n-1} \tilde{f}_1(r), \quad X_1 = (-1)^n X(r), \\ \tilde{f}_1(r) &= r^{2(n-1)} + r^{2(n-2)} \sum_{j>1} x_j^2 + r^{2(n-3)} \sum_{j,k>1} x_j^2 x_k^2 + \cdots + \prod_{j>1} x_j^2, \\ X(r) &= g^2 r^{2n} + \cdots. \end{aligned} \quad (3.177)$$

Now from (3.121) and (3.122), one has for both (3.173) and (3.175),

$$\mathcal{A} = \mathcal{A}_1 + \frac{\tilde{f}_1 dr}{X}. \quad (3.178)$$

As a result, for both even and odd dimensions ($i \leq n-1$),

$$h_v = r^{2(n-1)}, \quad h_i = r^{2(n-1-i)}, \quad h_{\mathcal{A}} = 0. \quad (3.179)$$

From (3.176),

$$\begin{aligned} \mathcal{A}_{i \neq 1} &= dt - \frac{h_v}{h_1} d\phi_1 + \sum_{j \neq 1, i} x_j^2 \left[\left(d\phi_1 - \frac{h_1}{h_v} dt \right) - r^2 \left(d\phi_2 - \frac{h_2}{h_v} dt \right) \right] \\ &\quad + \cdots + \prod_{j \neq 1, i} x_j^2 \left[\left(d\phi_{n-2} - \frac{h_{n-2}}{h_v} dt \right) - r^2 \left(d\phi_{n-1} - \frac{h_{n-1}}{h_v} dt \right) \right]. \end{aligned} \quad (3.180)$$

In odd dimensions, we also have

$$\begin{aligned} \mathcal{A}_n &= dt + \sum_{i=1}^n x_i^2 d\phi_1 + \sum_{i,j=1}^n x_i^2 x_j^2 d\phi_2 + \cdots + \prod_{i=1}^n x_i^2 d\phi_n, \\ &= dt + \sum_{j \neq 1, i} x_j^2 d\phi_1 + \sum_{j,k \neq 1, i} x_j^2 x_k^2 d\phi_2 + \cdots + \prod_{j \neq 1, i} x_j^2 d\phi_{n-1} \\ &\quad - r^2 \left(d\phi_1 + \sum_{j \neq 1, i} x_j^2 d\phi_2 + \cdots + \prod_{j \neq 1, i} x_j^2 d\phi_n \right) \\ &= dt - r^2 d\phi_1 + \sum_{j \neq 1, i} x_j^2 (d\phi_1 - r^2 d\phi_2) + \cdots \\ &\quad + \prod_{j \neq 1, i} x_j^2 \left(d\phi_{n-1} - r^2 d\phi_n \right) \end{aligned}$$

$$\begin{aligned}
&= dt - \frac{h_v}{h_1} d\phi_1 + \sum_{j \neq 1, i} x_j^2 \left[\left(d\phi_1 - \frac{h_1}{h_v} dt \right) - r^2 \left(d\phi_2 - \frac{h_2}{h_v} dt \right) \right] \\
&\quad + \cdots + \prod_{j \neq 1, i} x_j^2 \left[\left(d\phi_{n-1} - \frac{h_{n-1}}{h_v} dt \right) - r^2 \left(d\phi_n - \frac{h_n}{r^2 h_v} dt \right) \right]. \quad (3.181)
\end{aligned}$$

So it is obvious that both (3.173) and (3.175) can be put into the form of (3.127), with $h_{\mathcal{A}} = h_{\chi}^a = h_{tt} = 0$.

As mentioned at the beginning this example section, the Kerr/CFT correspondence is guaranteed to work if the metric can be put into the form of (3.127) or (3.128). Since all for all the examples studied in this subsection, the metrics can be put into the form of (3.127), the Kerr/CFT correspondence works for all the examples studied here.

b. Extremal Static Black Holes in Supergravity

The rotation plays a vital role in the calculation of the Kerr/CFT correspondence. However, since one can obtain the correct entropy for static black holes by taking a limit of rotating black holes, this suggests that there may be an alternative strictly static description that is not singular. One way to achieve this is based upon the observation that static charged black holes in many gauged supergravities can be lifted, by means of consistent Kaluza-Klein reduction formulae derived in [142], to become solutions in the ten or eleven-dimensional supergravities that arise as the low-energy limits of string theory or M-theory. The electric charges of the static black holes acquire the interpretation of rotations in the internal (spherical) dimensions, after the lifting has been performed. The procedure developed in the previous subsection can then be applied to the lifted solutions. This is done in [25] with Hong Lü, Chris Pope and Justin Vazquez-Poritz.

Here we revisit the examples from the perspective of using (3.127). For this

purpose, we start with the various reduction ansatz given in [142].

- For the S^5 reduction of type IIB supergravity, the ansatz for the ten-dimensional metric is

$$ds_{10}^2 = \sqrt{\widetilde{\Delta}} ds_5^2 + \frac{1}{g^2 \sqrt{\widetilde{\Delta}}} \sum_{i=1}^3 X_i^{-1} \left[d\mu_i^2 + \mu_i^2 (d\phi_i + g A^i)^2 \right], \quad (3.182)$$

where $X_1 X_2 X_3 = 1$.

- For the S^7 reduction of $D = 11$ supergravity, the ansatz for the eleven-dimensional metric is

$$ds_{11}^2 = \widetilde{\Delta}^{2/3} ds_4^2 + g^{-2} \widetilde{\Delta}^{-1/3} \sum_i X_i^{-1} \left[d\mu_i^2 + \mu_i^2 (d\phi_i + g A_{(1)}^i)^2 \right], \quad (3.183)$$

where $\widetilde{\Delta} = \sum_{i=1}^4 X_i \mu_i^2$, and $\sum_i \mu_i^2 = 1$ and $X_1 X_2 X_3 X_4 = 1$.

- For the S^4 reduction of $D = 11$ supergravity, the ansatz for the eleven-dimensional metric is

$$\begin{aligned} ds_{11}^2 = & \widetilde{\Delta}^{1/3} ds_7^2 + g^{-2} \widetilde{\Delta}^{-2/3} \left\{ X_0^{-1} d\mu_0^2 \right. \\ & \left. + \sum_{i=1}^2 X_i^{-1} \left[d\mu_i^2 + \mu_i^2 (d\phi_i + g A_{(1)}^i)^2 \right] \right\}, \end{aligned} \quad (3.184)$$

where $\widetilde{\Delta} = \sum_{\alpha=0}^2 X_\alpha \mu_\alpha^2$ with $\mu_0^2 + \mu_1^2 + \mu_2^2 = 1$, and the auxiliary variable $X_0 \equiv (X_1 X_2)^{-2}$.

- For the S^4 reduction of type IIA supergravity, the ansatz for the ten-dimensional metric is found in [143],

$$\begin{aligned} d\hat{s}_{10}^2 = & (\sin \xi)^{\frac{1}{12}} X^{\frac{1}{8}} \left[\Delta^{\frac{3}{8}} ds_6^2 + 2g^{-2} \Delta^{\frac{3}{8}} X^2 d\xi^2 \right. \\ & \left. + \frac{1}{2} g^{-2} \Delta^{-\frac{5}{8}} X^{-1} \cos^2 \xi \sum_{i=1}^3 (\sigma^i + g A_{(1)}^i)^2 \right], \end{aligned} \quad (3.185)$$

where $X = e^{-\frac{1}{2\sqrt{2}}\phi}$, and $\Delta = X \cos^2 \xi + X^{-3} \sin^2 \xi$. The quantities σ^i are left-invariant 1-forms on S^3 , which satisfy $d\sigma^i = -\frac{1}{2}\epsilon_{ijk} \sigma^j \wedge \sigma^k$. One can parameterize them as

$$\sigma_1 = d\theta, \quad \sigma_2 = \sin^2 \theta d\phi, \quad \sigma_3 = d\psi + \cos \theta d\phi. \quad (3.186)$$

For all the examples that will be discussed in the following, the lower dimension metrics will be static. So the metric will not have any cross terms involving $d\hat{t}$ and the azimuthal angles. So for the terms involved in (3.121) and (3.122), one will have $f_a = 0$. What's more, all the gauge fields are of the particular form, $A^i = \Phi^i(r)d\hat{t}$; and for (3.185), only $A_{(1)}^3 \neq 0$. So it is easy to see that $h_i/h_v = -g\Phi^i(r)$. It is then obvious that all the metrics (3.182), (3.183), (3.184), and (3.185) will be of the form (3.127). Now let's look at explicit examples.

The first example is with the maximal gauged supergravity in $D = 5$. It has $SO(6)$ gauge symmetry. The Cartan subgroup is $U(1)^3$. The five-dimensional three-charge static AdS black hole solution was constructed in [98]. We adopt the convention of [142], and the solution is given by

$$\begin{aligned} ds_5^2 &= -\mathcal{H}^{-2/3} f d\hat{t}^2 + \mathcal{H}^{1/3} (f^{-1} d\hat{r}^2 + \hat{r}^2 d\Omega_{3,\epsilon}^2), \\ X_i &= H_i^{-1} \mathcal{H}^{1/3}, \quad A_{(1)}^i = \Phi_i d\hat{t}, \quad \Phi_i = -(1 - H_i^{-1})\alpha_i, \\ f &= \epsilon - \frac{\mu}{\hat{r}^2} + g^2 \hat{r}^2 \mathcal{H}, \quad \mathcal{H} = H_1 H_2 H_3, \quad H_i = 1 + \frac{\ell_i^2}{\hat{r}^2}, \\ \alpha_i &= \frac{\sqrt{1 + \epsilon \sinh^2 \beta_i}}{\sinh \beta_i}, \quad \ell_i^2 = \mu \sinh^2 \beta_i, \end{aligned} \quad (3.187)$$

where $d\Omega_{3,\epsilon}^2$ is the unit metric for S^3 , T^3 or H^3 for $\epsilon = 1, 0$ or -1 , respectively. If all the charge parameters β_i are set equal, the solution becomes the five-dimensional

Reissner-Nordström AdS black hole. We see that

$$\begin{aligned} \frac{h_i}{h_v} &= -g\Phi_i, \quad h_\chi^\phi = h_{tt} = 0, \\ \mathcal{A} = d\hat{t} + \frac{\sqrt{\mathcal{H}}}{f} dr &\implies h_v = \sqrt{\mathcal{H}}, \quad f_i = 0, \quad h_{\mathcal{A}} = 0. \end{aligned} \quad (3.188)$$

The second example is with the maximum gauged supergravity in $D = 4$. It has $SO(8)$ gauge group, with the Cartan subgroup $U(1)^4$. The four-charge static AdS black hole was constructed in [144, 145]. Following the convention of [142], the four-dimensional 4-charge AdS black hole solution is given by

$$\begin{aligned} ds_4^2 &= -\mathcal{H}^{-1/2} f d\hat{t}^2 + \mathcal{H}^{1/2} (f^{-1} d\hat{r}^2 + \hat{r}^2 d\Omega_{2,\epsilon}^2), \\ X_i &= H_i^{-1} \mathcal{H}^{1/4}, \quad A_{(1)}^i = \Phi_i d\hat{t}, \quad \Phi_i = -(1 - H_i^{-1})\alpha_i, \\ f &= \epsilon - \frac{\mu}{\hat{r}} + 4g^2 \hat{r}^2 \mathcal{H}, \quad \mathcal{H} = H_1 H_2 H_3 H_4, \quad H_i = 1 + \frac{\ell_i}{\hat{r}}, \\ \alpha_i &= \frac{\sqrt{1 + \epsilon \sinh^2 \beta_i}}{\sinh \beta_i}, \quad \ell_i = \mu \sinh^2 \beta_i, \end{aligned} \quad (3.189)$$

where $d\Omega_{2,\epsilon}^2$ is the unit metric for S^2 , T^2 or H^2 for $\epsilon = 1, 0$ or -1 , respectively. If the charge parameters β_i are set equal, the solution becomes the standard Reissner-Nordström AdS black hole. We see that

$$\begin{aligned} \frac{h_i}{h_v} &= -g\Phi_i, \quad h_\chi^\phi = h_{tt} = 0, \\ \mathcal{A} = d\hat{t} + \frac{\sqrt{\mathcal{H}}}{f} dr &\implies h_v = \sqrt{\mathcal{H}}, \quad f_i = 0, \quad h_{\mathcal{A}} = 0. \end{aligned} \quad (3.190)$$

The third example is with the maximal gauged supergravity in $D = 7$. It has $SO(5)$ gauge symmetry, whose Cartan subgroup is $U(1)^2$. The seven-dimensional 2-charge AdS black hole solution is given by [142]

$$\begin{aligned} ds_7^2 &= -\mathcal{H}^{-4/5} f d\hat{t}^2 + \mathcal{H}^{1/5} (f^{-1} d\hat{r}^2 + \hat{r}^2 d\Omega_{5,\epsilon}^2), \\ X_i &= H_i^{-1} \mathcal{H}^{2/5}, \quad A_{(1)}^i = \Phi_i d\hat{t}, \quad \Phi_i = -(1 - H_i^{-1})\alpha_i, \end{aligned}$$

$$\begin{aligned}
f &= \epsilon - \frac{\mu}{\hat{r}^4} + \frac{1}{4}g^2\hat{r}^2\mathcal{H}, \quad \mathcal{H} = H_1H_2, \quad H_i = 1 + \frac{\ell_i^4}{\hat{r}^4}, \\
\alpha_i &= \frac{\sqrt{1 + \epsilon \sinh^2 \beta_i}}{\sinh \beta_i}, \quad \ell_i^4 = \mu \sinh^2 \beta_i,
\end{aligned} \tag{3.191}$$

where $d\Omega_{5,\epsilon}^2$ is the unit metric for S^5 , T^5 or H^5 for $\epsilon = 1, 0$ or -1 , respectively. We see that

$$\begin{aligned}
\frac{h_i}{h_v} &= -g\Phi_i, \quad h_\chi^\phi = h_{tt} = 0, \\
\mathcal{A} = d\hat{t} + \frac{\sqrt{\mathcal{H}}}{f}dr &\implies h_v = \sqrt{\mathcal{H}}, \quad f_i = 0, \quad h_{\mathcal{A}} = 0.
\end{aligned} \tag{3.192}$$

The last example is with the gauged supergravity in $D = 6$ constructed in [146]. It has a $SU(2)$ gauge symmetry. The $U(1)$ charged AdS black hole was constructed in [143],

$$\begin{aligned}
ds_6^2 &= -H^{-3/2}f d\hat{t}^2 + H^{1/2}(f^{-1}d\hat{r}^2 + \hat{r}^2 d\Omega_{4,\epsilon}^2), \\
X &= H^{-1/4}, \quad A_{(1)} = \Phi d\hat{t}, \quad \Phi = -\sqrt{2}(1 - H^{-1})\alpha d\hat{t}, \\
f &= \epsilon - \frac{\mu}{\hat{r}^3} + \frac{2}{9}g^2\hat{r}^2H^2, \quad H = 1 + \frac{\ell^3}{\hat{r}^3}, \\
\alpha &= \frac{\sqrt{1 + \epsilon \sinh^2 \beta}}{\sinh \beta}, \quad \ell^3 = \mu \sinh^2 \beta.
\end{aligned} \tag{3.193}$$

We see that

$$\begin{aligned}
\frac{h_{\sigma^3}}{h_v} &= -g\Phi, \quad h_{\sigma^1} = h_{\sigma^2} = h_\chi^\phi = h_{tt} = 0, \\
\mathcal{A} = d\hat{t} + \frac{H}{f}dr &\implies h_v = H, \quad f_i = 0, \quad h_{\mathcal{A}} = 0.
\end{aligned} \tag{3.194}$$

As mentioned at the beginning this example section, the Kerr/CFT correspondence is guaranteed to work if the metric can be put into the form of (3.127) or (3.128). Since all for all the examples studied in this subsection, the metrics can be put into the form of (3.127), the Kerr/CFT correspondence works for all the examples

studied here.

c. Extremal Rotating Black Holes in Supergravity

The Kerr/CFT correspondence for rotating black hole solutions in supergravity theories were studied in [134]. There the significance of the universal form of the near-horizon metric (3.147) was first discovered. Here, we shall repeat some of the examples, just to show the general applicability of the metric (3.127) and (3.128).

In the five dimensional (un)gauged supergravities, there are three non-extremal solutions that cannot accommodate each other.⁷ They are the three-charge two-rotation Cvetič-Youm solution [96] in the ungauged supergravity, the three-charge equal-rotation solution [100] and the three-charge (two of which equal) two-rotation solution [21] in the gauged supergravity.

The Cvetič-Youm solution is given in (2.38), which we copy here for easy reference,

$$\begin{aligned}
ds^2 &= (H_1 H_2 H_3)^{1/3} \left[\frac{dx^2}{4X} + \frac{dy^2}{4Y} + \frac{U}{G} \left(d\chi - \frac{Z}{U} d\sigma \right)^2 + \frac{XY}{U} d\sigma^2 \right] \\
&\quad - \frac{G \left(dt + \tilde{\mathcal{A}} \right)^2}{(H_1 H_2 H_3)^{2/3}}, \\
\tilde{\mathcal{A}} &= \frac{2mc_1 c_2 c_3 [(a^2 + b^2 - y)d\sigma - abd\chi]}{x + y - 2m} - \frac{2ms_1 s_2 s_3 (abd\sigma - yd\chi)}{x + y}, \\
X &= (x + a^2)(x + b^2) - 2mx, \quad Y = -(a^2 - y)(b^2 - y), \\
U &= yX - xY, \quad Z = ab(X + Y), \quad G = (x + y)(x + y - 2m), \\
\mathcal{A}_i &= \frac{2m}{H_i} \left\{ c_i s_i dt + s_i c_j c_k \left[abd\chi + (y - a^2 - b^2)d\sigma \right] \right. \\
&\quad \left. + c_i s_j s_k (abd\sigma - yd\chi) \right\}, \quad i \neq j \neq k,
\end{aligned}$$

⁷As far as solutions are concerned, one can go from a gauged supergravity to its ungauged counterpart by simply turning off the gauge coupling constant, which is equivalent to the cosmological constant.

$$X_i = \frac{H_1^{1/3} H_2^{1/3} H_3^{1/3}}{H_i}, \quad H_i = x + y + 2ms_i^2, \quad (3.195)$$

where $s_i = \sinh \delta_i$, $c_i = \cosh \delta_i$ and $i, j, k = 1, 2, 3$. The variables χ and σ are related to the canonical azimuthal angles by

$$\sigma = \frac{a\hat{\phi}_1 - b\hat{\phi}_2}{a^2 - b^2}, \quad \chi = \frac{b\hat{\phi}_1 - a\hat{\phi}_2}{a^2 - b^2}. \quad (3.196)$$

Near the horizon, σ is playing the role of the time direction as in the Schwarzschild solution. We have for (3.121) and (3.122),

$$\mathcal{A} = d\sigma + \frac{(a^2 - b^2)\sqrt{x} dx}{2X} \sqrt{1 - \frac{yX}{xY}}. \quad (3.197)$$

By comparing various terms, we find that

$$\begin{aligned} h_v &= \frac{ab(c_1^2 c_2^2 c_3^2 + s_1^2 s_2^2 s_3^2) - (a^2 + b^2 - 2m)c_1 c_2 c_3 s_1 s_2 s_3}{abc_1 c_2 c_3 + x s_1 s_2 s_3} m \sqrt{x}, \\ h_1 &= \frac{a(b^2 + x)s_1 s_2 s_3 - b(b^2 - 2m + x)c_1 c_2 c_3}{2(abc_1 c_2 c_3 + x s_1 s_2 s_3)} \sqrt{x}, \\ h_2 &= \frac{b(a^2 + x)s_1 s_2 s_3 - a(a^2 - 2m + x)c_1 c_2 c_3}{2(abc_1 c_2 c_3 + x s_1 s_2 s_3)} \sqrt{x}, \end{aligned} \quad (3.198)$$

and so

$$\begin{aligned} d\sigma &= \frac{a}{a^2 - b^2} d\hat{\phi}_1 - \frac{b}{a^2 - b^2} d\hat{\phi}_2, \\ -\frac{U}{4Y} &= \left(\frac{a}{a^2 - b^2} h_1 - \frac{b}{a^2 - b^2} h_2 \right)^2 - \frac{yX}{4Y}, \\ d\chi - \frac{Z}{U} d\sigma &= -\frac{\left(\frac{x}{x+y} + \frac{a^2}{x+y-2m} \right) (a^2 - y)b}{\left(\frac{xy}{x+y} + \frac{a^2 b^2}{x+y-2m} \right) (a^2 - b^2)} \left(d\hat{\phi}_1 - \frac{h_1}{h_v} dt \right) \\ &\quad + \frac{\left(\frac{x}{x+y} + \frac{b^2}{x+y-2m} \right) (b^2 - y)a}{\left(\frac{xy}{x+y} + \frac{a^2 b^2}{x+y-2m} \right) (a^2 - b^2)} \left(d\hat{\phi}_2 - \frac{h_2}{h_v} dt \right) \\ &\quad - \frac{\left(\frac{abs_1 s_2 s_3}{x+y-2m} - \frac{c_1 c_2 c_3 y}{x+y} \right) X \sqrt{x} dt}{2h_v \left(\frac{xy}{x+y} + \frac{a^2 b^2}{x+y-2m} \right) (abc_1 c_2 c_3 + s_1 s_2 s_3 x)}, \\ dt + \tilde{\mathcal{A}} &= \frac{2m(a^2 - y)}{a^2 - b^2} \left(\frac{ac_1 c_2 c_3}{x + y - 2m} - \frac{bs_1 s_2 s_3}{x + y} \right) \left(d\hat{\phi}_1 - \frac{h_1}{h_v} dt \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{2m(b^2 - y)}{a^2 - b^2} \left(\frac{as_1s_2s_3}{x + y} - \frac{bc_1c_2c_3}{x + y - 2m} \right) \left(d\hat{\phi}_2 - \frac{h_2}{h_v} dt \right) \\
& + \frac{2m^2 X \sqrt{x} c_1c_2c_3s_1s_2s_3 dt}{h_v(abc_1c_2c_3 + s_1s_2s_3x)(x + y - 2m)(x + y)}. \quad (3.199)
\end{aligned}$$

It is obvious that (3.195) is of the form (3.128) with $h_{\mathcal{A}}, h_{\chi}^1, h_{\chi}^2 \neq 0$ but $h_{tt} = 0$. As a side remark, note the gauge fields can be written as

$$\begin{aligned}
\mathcal{A}_i = & \frac{2m}{(a^2 - b^2)h_i} \left\{ (bc_i s_j s_k - as_i c_j c_k)(a^2 - y) \left(d\hat{\phi}_1 - \frac{h_1}{h_v} dt \right) \right. \\
& + (bs_i c_j c_k - ac_i s_j s_k)(b^2 - y) \left(d\hat{\phi}_2 - \frac{h_2}{h_v} dt \right) \Big\} \\
& + \frac{abc_i s_i (c_j^2 c_k^2 + s_j^2 s_k^2) - c_j c_k s_j s_k [x + c_i^2 (a^2 + b^2 - 2m)]}{(abc_i c_j c_k + s_i s_j s_k x) h_v / (m\sqrt{x})} dt \\
& + \frac{c_j c_k s_j s_k X m \sqrt{x}}{(abc_i c_j c_k + s_i s_j s_k x) h_i h_v} dt, \quad i \neq j \neq k. \quad (3.200)
\end{aligned}$$

When transforming to the coordinates on the horizon by (3.124), only the third line will lead to a divergence, but which can be absorbed as pure gauge.

For the three-charge equal-rotation solution in the gauged supergravity [100], the result is given in (2.41). Here we shall use the original form as in [100],

$$\begin{aligned}
ds^2 = & R \left\{ -\frac{X}{f_1} dt^2 + \frac{r^2}{X} dr^2 + d\theta^2 + \cos^2 \theta \sin^2 \theta (d\phi - d\psi)^2 \right. \\
& \left. + \frac{f_1}{R^3} \left(\cos^2 \theta d\phi + \sin^2 \theta d\psi - \frac{f_2}{f_1} dt \right)^2 \right\}, \\
X = & r^4 - 2m(r^2 - \ell^2) + g^2 f_1, \quad f_1 = 2m\ell^2(r^2 + 2m\tilde{s}) + R^3, \\
f_2 = & 2m\ell r^2(c_1c_2c_3 - s_1s_2s_3) + 4m^2\ell s_1s_2s_3, \\
R = & (H_1H_2H_3)^{1/3}, \quad H_i = r^2 + 2ms_i^2, \quad i = 1, 2, 3, \\
\tilde{s} = & 2s_1s_2s_3(c_1c_2c_3 - s_1s_2s_3) - s_1^2s_2^2 - s_1^2s_3^2 - s_2^2s_3^2, \\
\mathcal{A}_i = & \frac{2m}{h_i} \left[c_i s_i dt + \ell(c_i s_j s_k - s_i c_j c_k)(\cos^2 \theta d\phi + \sin^2 \theta d\psi) \right]. \quad (3.201)
\end{aligned}$$

It is easy to tell that the metric is of the (3.127) with

$$h_v = r\sqrt{f_1}, \quad h_\phi = h_\psi = \frac{rf_2}{\sqrt{f_1}}, \quad h_{\mathcal{A}} = h_\chi^\phi = h_\chi^\psi = h_{tt} = 0. \quad (3.202)$$

After using (3.124), the gauge fields are also regular on the horizon up to some divergence which can be absorbed as pure gauge.

The three-charge (two of which equal) two-rotation solution in the gauged supergravity was found in [21], and the solution has been given in (2.107)-(2.111). Again, we copy some of the result here for easy reference,

$$\begin{aligned} ds^2 &= H_1^{2/3} H_3^{1/3} \left\{ (x^2 - y^2) \left(\frac{dx^2}{X} - \frac{dy^2}{Y} \right) - \frac{x^2 X (dt + y^2 d\sigma)^2}{(x^2 - y^2) f H_1^2} \right. \\ &\quad \left. + \frac{y^2 Y [dt + (x^2 + 2ms_1^2) d\sigma]^2}{(x^2 - y^2)(\gamma + y^2) H_1^2} \right. \\ &\quad \left. - U \left(dt + y^2 d\sigma + \frac{(x^2 - y^2) f H_1 [abd\sigma + (\gamma + y^2) d\chi]}{ab(x^2 - y^2) H_3 - 2ms_3 c_3 (\gamma + y^2)} \right)^2 \right\}, \\ \mathcal{A}^1 &= \mathcal{A}^2 = \frac{2ms_1 c_1 (dt + y^2 d\sigma)}{(x^2 - y^2) H_1}, \\ \mathcal{A}^3 &= \frac{2m \{s_3 c_3 (dt + y^2 d\sigma) - (s_1^2 - s_3^2) [abd\sigma + (\gamma + y^2) d\chi]\}}{(x^2 - y^2) H_3}, \\ f &= x^2 + \gamma + 2ms_3^2, \quad \gamma = 2abs_3 c_3 + (a^2 + b^2) s_3^2, \\ H_1 &= 1 + \frac{2ms_1^2}{x^2 - y^2}, \quad H_3 = 1 + \frac{2ms_3^2}{x^2 - y^2}. \end{aligned} \quad (3.203)$$

Comparing with (3.121) and (3.122), we see that

$$\begin{aligned} \mathcal{A} &= dt + y^2 d\sigma + \frac{(x^2 - y^2) \sqrt{f} H_1}{x X} dx \\ &= dt + y^2 d\sigma + \frac{(x^2 - y^2 + 2ms_1^2) \sqrt{f}}{x X} dx, \\ \Rightarrow h_v &= \frac{(x^2 + 2ms_1^2) \sqrt{f}}{x}, \quad h_\sigma = -\frac{\sqrt{f}}{x}. \end{aligned} \quad (3.204)$$

As a result,

$$dt + (x^2 + 2ms_1^2) d\sigma \propto d\sigma - \frac{h_\sigma}{h_v} dt, \quad (3.205)$$

and with $h_\chi = \frac{ab + 2mc_3s_3}{x\sqrt{f}}$,

$$\begin{aligned}
& dt + y^2 d\sigma + \frac{(x^2 - y^2)fH_1[abd\sigma + (\gamma + y^2)d\chi]}{ab(x^2 - y^2)H_3 - 2ms_3c_3(\gamma + y^2)} \\
&= \left\{ x + 2ms_1^2 + \frac{(ab + 2mc_3s_3)(x^2 - y^2)H_1(y^2 + \gamma)}{ab(x^2 - y^2)H_3 - 2mc_3s_3(y^2 + \gamma)} \right\} \left(d\sigma - \frac{h_\sigma}{h_v} dt \right) \\
&+ \frac{(y^2 + \gamma)(x^2 - y^2)fH_1}{ab(x^2 - y^2)H_3 - 2mc_3s_3(y^2 + \gamma)} \left(d\chi - \frac{h_\chi}{h_v} dt \right). \tag{3.206}
\end{aligned}$$

Now it is obvious that the metric in (3.203) is of the form (3.127). For the gauge fields, one has

$$\begin{aligned}
\mathcal{A}_1 &= \mathcal{A}_2 = \frac{2mc_1s_1y^2}{(x^2 - y^2)H_1} \left(d\sigma - \frac{h_\sigma}{h_v} dt \right) + \frac{2mc_1s_1}{x^2 + 2ms_1^2} dt, \\
\mathcal{A}_3 &= -\frac{2m}{(x^2 - y^2)H_3} \left\{ \left[ab(s_1^2 - s_3^2) - c_3s_3y^2 \right] \left(d\sigma - \frac{h_\sigma}{h_v} dt \right) \right. \\
&\quad \left. + (s_1^2 - s_3^2)(y^2 + \gamma) \left(d\chi - \frac{h_\chi}{h_v} dt \right) \right\} \\
&\quad + \frac{2m \left[c_3s_3f + (ab + 2mc_3s_3)(s_1^2 - s_3^2) \right]}{f(x^2 + 2ms_1^2)} dt. \tag{3.207}
\end{aligned}$$

Again, when (3.124) is used, the divergent pieces can be absorbed as pure gauge.

In the following, we give a few more solutions in dimensions other than five. Again, all these have been studied in [134]. We include them here just to show the general applicability of the metric (3.127) and (3.128).

The first example is the four-charge black hole of the ungauged supergravity in four dimension [147, 103],

$$ds_4^2 = -\frac{\rho^2 - 2m\hat{r}}{W} (d\hat{t} + B d\hat{\phi})^2 + W \left(\frac{d\hat{r}^2}{\Delta} + d\theta^2 + \frac{\Delta \sin^2 \theta d\hat{\phi}^2}{\rho^2 - 2m\hat{r}} \right). \tag{3.208}$$

The detail of various functions can be found in [134]. Notably,

$$\Delta = \hat{r}^2 - 2m\hat{r} + a^2, \quad \rho^2 = \hat{r}^2 + a^2 \cos^2 \theta, \quad W = W(r),$$

$$B = \frac{2ma^2 \sin^2 \theta [\hat{r} c_1 c_2 c_3 c_4 - (\hat{r} - 2m) s_1 s_2 s_3 s_4]}{a(\rho^2 - 2m\hat{r})}. \quad (3.209)$$

Note $\rho^2 - 2m\hat{r} = \Delta - a^2 \sin^2 \theta$. So when it comes close to the horizon, $d\hat{\phi}$ replaces $d\hat{t} + B d\hat{\phi}$ and become the time direction. What's more,

$$\begin{aligned} B &= -\frac{1}{B_0} \left(1 + \frac{\Delta}{a^2 \sin^2 \theta} \right) + \mathcal{O}(\Delta^2), \\ B_0 &= \frac{a}{2m[\hat{r} c_1 c_2 c_3 c_4 - (\hat{r} - 2m) s_1 s_2 s_3 s_4]}. \end{aligned} \quad (3.210)$$

Comparing (3.208) with (3.121), we have for (3.122),

$$\begin{aligned} \mathcal{A} &= d\hat{\phi} + \frac{\sqrt{a^2 \sin^2 \theta - \Delta}}{\Delta \sin \theta} d\hat{r} \\ &\approx d\hat{\phi} + \frac{a}{\Delta} d\hat{r} - \frac{d\hat{r}}{2a \sin^2 \theta}, \\ \implies h_{\hat{\phi}} &= a, \quad h_{\mathcal{A}} = -\frac{1}{2a \sin^2 \theta}. \end{aligned} \quad (3.211)$$

By letting $h_v = \frac{a}{B_0}$ and $h_{\chi}^{\hat{\phi}} = -\frac{1}{a \sin^2 \theta}$, we also have

$$d\hat{t} + B d\hat{\phi} \propto d\hat{\phi} - \frac{h_{\hat{\phi}} + h_{\chi}^{\hat{\phi}} \Delta}{h_v} d\hat{t} + \mathcal{O}(\Delta^2). \quad (3.212)$$

So (3.208) is of the form (3.128) with $h_{tt} = 0$.

The next example is the rotating black hole solution in four-dimensional $U(1)^4$ gauged supergravity with the four $U(1)$ charges pairwise equal [103]. The metric is

$$\begin{aligned} ds^2 &= H \left[-\frac{R}{H^2(\hat{r}^2 + y^2)} \left(d\hat{t} - \frac{a^2 - y^2}{\Xi a} d\hat{\phi} \right)^2 + \frac{\hat{r}^2 + y^2}{R} d\hat{r}^2 + \frac{\hat{r}^2 + y^2}{Y} dy^2 \right. \\ &\quad \left. + \frac{Y}{H^2(\hat{r}^2 + y^2)} \left(d\hat{t} - \frac{(\hat{r} + q_1)(\hat{r} + q_2) + a^2}{\Xi a} d\hat{\phi} \right)^2 \right], \end{aligned} \quad (3.213)$$

where

$$\begin{aligned} R &= \hat{r}^2 + a^2 + g^2(\hat{r} + q_1)(\hat{r} + q_2)[(\hat{r} + q_1)(\hat{r} + q_2) + a^2] - 2m\hat{r}, \\ Y &= (1 - g^2 y^2)(a^2 - y^2), \quad \Xi = 1 - g^2 a^2, \end{aligned}$$

$$H = \frac{(\hat{r} + q_1)(\hat{r} + q_2) + y^2}{\hat{r}^2 + y^2}, \quad q_I = 2ms_I^2, \quad s_I = \sinh \delta_I. \quad (3.214)$$

Comparing (3.213) with (3.121), we have for (3.122),

$$\begin{aligned} \mathcal{A} &= d\hat{t} - \frac{a^2 - y^2}{\Xi a} d\hat{\phi} + \frac{(\hat{r} + q_1)(\hat{r} + q_2) + y^2}{R} d\hat{r}, \\ \Rightarrow h_v &= \frac{(\hat{r} + q_1)(\hat{r} + q_2) + a^2}{R}, \quad h_{\hat{\phi}} = \frac{\Xi a}{R}. \end{aligned} \quad (3.215)$$

It is easy to see that

$$d\hat{t} - \frac{(\hat{r} + q_1)(\hat{r} + q_2) + a^2}{\Xi a} d\hat{\phi} \propto d\hat{\phi} - \frac{h_{\hat{\phi}}}{h_v} d\hat{t}. \quad (3.216)$$

So (3.213) is of the form (3.127) with $h_{\mathcal{A}} = h_{\hat{\chi}}^{\hat{\phi}} = h_{tt} = 0$.

A single-charge two-rotation solution to the six-dimensional SU(2) gauged supergravity was found in [106]. The metric is

$$\begin{aligned} ds^2 &= H^{1/2} \left\{ -\frac{R}{H^2 U} \tilde{\mathcal{A}}^2 + \frac{(\hat{r}^2 + y^2)(y^2 - z^2)}{Y} dy^2 + \frac{Y \tilde{\mathcal{A}}_Y^2}{(\hat{r}^2 + y^2)(y^2 - z^2)} \right. \\ &\quad \left. + \frac{U}{R} d\hat{r}^2 + \frac{(\hat{r}^2 + z^2)(z^2 - y^2)}{Z} dz^2 + \frac{Z \tilde{\mathcal{A}}_Z^2}{(\hat{r}^2 + z^2)(z^2 - y^2)} \right\}, \end{aligned} \quad (3.217)$$

$$\begin{aligned} \tilde{\mathcal{A}}_Y &= d\hat{t} - (\hat{r}^2 + a^2)(a^2 - z^2) \frac{d\hat{\phi}_1}{\epsilon_1} - (\hat{r}^2 + b^2)(b^2 - z^2) \frac{d\hat{\phi}_2}{\epsilon_2} - \frac{q\hat{r}\tilde{\mathcal{A}}}{HU}, \\ \tilde{\mathcal{A}}_Z &= d\hat{t} - (\hat{r}^2 + a^2)(a^2 - y^2) \frac{d\hat{\phi}_1}{\epsilon_1} - (\hat{r}^2 + b^2)(b^2 - y^2) \frac{d\hat{\phi}_2}{\epsilon_2} - \frac{q\hat{r}\tilde{\mathcal{A}}}{HU}, \end{aligned} \quad (3.218)$$

where the various functions and constants can be found in [134]. The ones relevant for us are

$$\begin{aligned} U &= (\hat{r}^2 + y^2)(\hat{r}^2 + z^2), \quad H = 1 + \frac{q\hat{r}}{U}, \\ \tilde{\mathcal{A}} &= d\hat{t} - (a^2 - y^2)(a^2 - z^2) \frac{d\hat{\phi}_1}{\epsilon_1} - (b^2 - y^2)(b^2 - z^2) \frac{d\hat{\phi}_2}{\epsilon_2}. \end{aligned} \quad (3.219)$$

Comparing (3.217) with (3.121), we have for (3.122),

$$\mathcal{A} = \tilde{\mathcal{A}} + \frac{HU}{R} dr. \quad (3.220)$$

By comparing various terms, one can find

$$\begin{aligned} h_v &= (\hat{r}^2 + a^2)(\hat{r}^2 + b^2) + q\hat{r}, \\ h_1 &= \frac{\hat{r}^2 + b^2}{a^2 - b^2} \epsilon_1, \quad h_2 = \frac{\hat{r}^2 + a^2}{b^2 - a^2} \epsilon_2, \end{aligned} \quad (3.221)$$

and

$$\begin{aligned} \tilde{\mathcal{A}}_Y &= \frac{(z^2 - a^2)[q\hat{r} + (\hat{r}^2 + a^2)(\hat{r}^2 + z^2)](\hat{r}^2 + y^2)}{HU\epsilon_1} \left(d\hat{\phi}_1 - \frac{h_1}{h_v} d\hat{t} \right) \\ &\quad + \frac{(z^2 - b^2)[q\hat{r} + (\hat{r}^2 + b^2)(\hat{r}^2 + z^2)](\hat{r}^2 + y^2)}{HU\epsilon_2} \left(d\hat{\phi}_2 - \frac{h_2}{h_v} d\hat{t} \right), \\ \tilde{\mathcal{A}}_Z &= \frac{(y^2 - a^2)[q\hat{r} + (\hat{r}^2 + a^2)(\hat{r}^2 + y^2)](\hat{r}^2 + z^2)}{HU\epsilon_1} \left(d\hat{\phi}_1 - \frac{h_1}{h_v} d\hat{t} \right) \\ &\quad + \frac{(y^2 - b^2)[q\hat{r} + (\hat{r}^2 + b^2)(\hat{r}^2 + y^2)](\hat{r}^2 + z^2)}{HU\epsilon_2} \left(d\hat{\phi}_2 - \frac{h_2}{h_v} d\hat{t} \right). \end{aligned} \quad (3.222)$$

So (3.217) is of the form (3.127) with $h_{\mathcal{A}} = h_{\hat{\chi}}^{\hat{\phi}} = h_{tt} = 0$.

The single-charge three-rotation black hole solution to the seven-dimensional SO(5) gauged supergravity was found in [105]. The metric is

$$\begin{aligned} ds^2 &= H^{2/5} \left\{ -\frac{R}{H^2 U} \tilde{\mathcal{A}}^2 + \frac{U}{R} d\hat{r}^2 + \frac{(\hat{r}^2 + y^2)(y^2 - z^2)}{Y} dy^2 \right. \\ &\quad \left. + \frac{(\hat{r}^2 + z^2)(z^2 - y^2)}{Z} dz^2 + \frac{Y \tilde{\mathcal{A}}_Y^2}{(\hat{r}^2 + y^2)(y^2 - z^2)} \right. \\ &\quad \left. + \frac{Z \tilde{\mathcal{A}}_Z^2}{(\hat{r}^2 + z^2)(z^2 - y^2)} + \frac{a_1^2 a_2^2 a_3^2}{\hat{r}^2 y^2 z^2} \tilde{\mathcal{A}}_7^2 \right\}, \\ \tilde{\mathcal{A}}_Y &= d\hat{t} - \sum_{i=1}^3 \frac{(\hat{r}^2 + a_i^2) \gamma_i}{a_i^2 - y^2} \frac{d\hat{\phi}_i}{\epsilon_i} - \frac{q}{HU} \tilde{\mathcal{A}}, \\ \tilde{\mathcal{A}}_Z &= d\hat{t} - \sum_{i=1}^3 \frac{(\hat{r}^2 + a_i^2) \gamma_i}{a_i^2 - z^2} \frac{d\hat{\phi}_i}{\epsilon_i} - \frac{q}{HU} \tilde{\mathcal{A}}, \end{aligned}$$

$$\tilde{\mathcal{A}}_7 = d\hat{t} - \sum_{i=1}^3 \frac{(\hat{r}^2 + a_i^2)\gamma_i}{a_i^2} \frac{d\hat{\phi}_i}{\epsilon_i} - \frac{q}{HU} \left(1 + \frac{gy^2z^2}{a_1a_2a_3}\right) \tilde{\mathcal{A}}, \quad (3.223)$$

where the various functions and constants can be found in [134]. The ones relevant for us are

$$\begin{aligned} U &= (\hat{r}^2 + y^2)(\hat{r}^2 + z^2), \quad \gamma_i = a_i^2(a_i^2 - y^2)(a_i^2 - z^2), \\ H &= 1 + \frac{q}{(\hat{r}^2 + y^2)(\hat{r}^2 + z^2)}, \quad \tilde{\mathcal{A}} = d\hat{t} - \sum_{i=1}^3 \gamma_i \frac{d\hat{\phi}_i}{\epsilon_i}. \end{aligned} \quad (3.224)$$

Comparing (3.223) with (3.121), we have for (3.122),

$$\mathcal{A} = \tilde{\mathcal{A}} + \frac{HU}{R} dr. \quad (3.225)$$

By comparing various terms, one can find

$$\begin{aligned} h_v &= \frac{(r^2 + a_1^2)(r^2 + a_2^2)(r^2 + a_3^2) + q(r^2 - ga_1a_2a_3)}{r^2}, \\ h_i &= \frac{a_i(r^2 + a_j^2)(r^2 + a_k^2) - gqa_ja_k}{a_i(a_i^2 - a_j^2)(a_i^2 - a_k^2)r^2} \epsilon_i, \quad i \neq j \neq k, \end{aligned} \quad (3.226)$$

and

$$\begin{aligned} \tilde{\mathcal{A}}_Y &= \sum_{i=1}^3 \frac{(z^2 - a_i^2)[q + (\hat{r}^2 + a_i^2)(\hat{r}^2 + z^2)](\hat{r}^2 + y^2)a_i^2}{HU\epsilon_i} \left(d\hat{\phi}_i - \frac{h_i}{h_v} d\hat{t}\right), \\ \tilde{\mathcal{A}}_Z &= \sum_{i=1}^3 \frac{(y^2 - a_i^2)[q + (\hat{r}^2 + a_i^2)(\hat{r}^2 + y^2)](\hat{r}^2 + z^2)a_i^2}{HU\epsilon_i} \left(d\hat{\phi}_i - \frac{h_i}{h_v} d\hat{t}\right), \\ \tilde{\mathcal{A}}_7 &= \sum_{i=1}^3 \frac{\gamma_i \left[\frac{q(a_1a_2a_3 + gy^2z^2)}{HU} - \frac{a_1a_2a_3}{a_i^2} (r^2 + a_i^2) \right]}{a_1a_2a_3\epsilon_i} \left(d\hat{\phi}_i - \frac{h_i}{h_v} d\hat{t}\right). \end{aligned} \quad (3.227)$$

So (3.223) is of the form (3.127) with $h_{\mathcal{A}} = h_{\chi}^{\hat{\phi}} = h_{tt} = 0$.

As mentioned at the beginning this example section, the Kerr/CFT correspondence is guaranteed to work if the metric can be put into the form of (3.127) or (3.128). Since all for all the examples studied in this subsection, the metrics can be

put into the form of either (3.127) or (3.128), the Kerr/CFT correspondence works for all the examples studied here.

C. Summary of the Chapter

In this chapter, we studied the calculation of the entropy for general extremal stationary and axisymmetric black holes, by using the Kerr/CFT correspondence first conjectured in [26].

To do that, we first discussed the construction of the conservation laws and the definition of conserved charges in curved spacetimes. We have also explained the treatment of asymptotic symmetries by using the covariant phase space method (particularly in the form used in [117, 130]) in great detail. This provided the necessary techniques needed in the calculation of the Kerr/CFT correspondence.

We then derived two general ansatz that can cover all the stationary and axisymmetric single black holes when it comes close to the horizon, based on the assumption that the horizon should not bear any intrinsic singularities — especially, the metric should be regular on the horizon with appropriately chosen coordinates. After this is done, we find that both ansatz lead to a unique form of the near-horizon metric, which was studied in [135, 136] for limited systems, as soon as the extremal limit is taken. Thus we show that the common form of the near-horizon metric is valid for all stationary and axisymmetric single black holes. Then, as was first discovered in [134], this common form of the near-horizon metric is all that we need to successfully calculate the Bekenstein-Hawking entropy by using the Kerr/CFT correspondence. In this work, we have proved this point with explicit analytical calculations. We hope this will provide a slightly new perspective to the Kerr/CFT correspondence.

Our calculation of the central charge (3.156) only contains contribution from

the Einstein-Hilbert action, so the match of the entropy from the CFT side to the Bekenstein-Hawking entropy suggests that the matter contribution to the central charge is zero [133, 141]. Apart from this, there is also an issue related to more complicated gravitational theories than that given by the Einstein-Hilbert action. It is known that the entropy will be modified. Some related work can be found in [139, 138].

In the later part of the chapter, we have also used a few examples to solicit some confidence on the two general ansatz that we have derived for stationary and axisymmetric single black holes when it comes close to the horizon.

Finally, note that there are limitations to the current understanding of the Kerr/CFT correspondence. First, the procedure can only work for extremal and near extremal black holes [63, 64]. Second, very little is known about the dual CFT. In higher dimensions, we find that each rotation will correspond to a copy of the Virasoro algebra, and hence an independent CFT. Each CFT is able to give the same entropy for the solution. However, apart from this, one knows nothing about the nature of the quantum states making up the system.

CHAPTER IV

SUMMARY

Currently the field of fundamental physics has already been organized into several well structured subregions. At the macroscopic level, we have General Relativity believed to work for cosmic scales. At the microscopic level, all the complexity of nature are believed to be governed by three fundamental interactions, all of which can be described by using gauged field theories. In the middle, there is statistics and thermodynamics action as a bridge, connecting the microscopic world to the macroscopic. In the whole picture, quantum theory and symmetry pose as the fundamental organizing principles. The role of symmetry is well appreciated both in General Relativity and in the field theories of particle physics. But the quantum theory only finds itself staying with the particle physics. A practical quantum theory of gravity is still out of reach.

In the search of the quantum gravity theory, black holes are playing a significant role. Due to the smallness of Newton's constant, the effect of quantum gravity is expected to be very small. But black holes are one of the few places where such effect can be important. Significant progress were made in the study of quantum gravity, when the black hole thermodynamics was discovered in the mid-1970's [6, 7, 8] and when the gauge/gravity duality was discovered in the later 1990's [13, 14, 15, 16]. With these, black holes are becoming even more important because they provide the necessary examples where various ideas can be tested.

Black hole solutions started to be found right after the construction of the theory of General Relativity [18]. Due to the development of the String theory and

supergravity theories, black hole solutions in dimensions other than four are also drawing much attention [71, 96]. Black hole solutions in the AdS black ground are also constructed [102, 88] for use in the AdS/CFT correspondence [13].

In this dissertation, we have constructed two set of solutions in two different theories.

The first is a Plebanski-Demianski type solution (2.45) in five dimensional pure Einstein gravity. The solution is then generalized to include electric and magnetic dipole charges (2.89). These solutions go back to some earlier solutions when appropriate limits are taken. With particular topologies at the horizon and at the spatial infinity, the general solutions also demonstrated the un-uniqueness of black hole solutions in higher dimensions, as compared to the no-hair theorem known in four dimensions.

The second is a three-charge (two of which equal) two rotation solution (2.107) to the five dimensional maximal supergravity. The general solutions here would have a total of six parameters, the mass, two rotations, and three electric charges supported by the three gauge fields in the $U(1)^3$ Cartan subgroup of $SO(6)$. They could equivalently be regarded as solutions in $\mathcal{N} = 2$ gauged supergravity coupled to two vector multiplets. Our solution covers most of what's already found in the theory [97]. There are only two other independent solutions left. The first is the three-charge two-rotation Cvetič-Youm solution [96] in the ungauged supergravity, and the second is the three-charge equal-rotation solution [100] in the gauged supergravity. It will be of great interest if the most general solution in this theory can be found.

The later part of the dissertation deals with the entropy of black holes. Inspired by the successful story of the study of the black body radiation at the end of the 19th century, people hope that the study of the black hole entropy may lead to the discovery of the quantum gravity theory. Significant progress has been made after

Maldacena discovered the AdS/CFT correspondence [13]. It was then realized that the gauge/gravity duality may be a general feature for the quantum gravity. The Kerr/CFT correspondence conjectured in [26] is one of such examples. This conjecture assumes that all the dynamical degrees of freedom of an extremal black hole reside on the horizon. What's more, one can identify the corresponding phase space with that of a two dimensional conformal field theory. The correction Bekenstein-Hawking entropy can then be calculated from the central charge of the CFT by using Cardy's formula.

In this dissertation, we proved the applicability of the Kerr/CFT correspondence to general extremal stationary and axisymmetric black hole. We did this by first deriving two general ansatz that can cover all the stationary and axisymmetric single black holes when it comes close to the horizon, based on the assumption that the metric should be intrinsically regular on the horizon. Then we find that both ansatz lead to a unique form of the near-horizon metric when the extremal limit is taken. Such form of the near-horizon metric was previously studied in [135, 136] for limited systems. Finally, we explicitly show that this form of the near-horizon metric is enough to support a successful calculation of the black hole entropy by using the Kerr/CFT correspondence. Part of the result has already been obtained in [134], but here we are doing it in a more general fashion.

There are limitations to the current understanding of the Kerr/CFT correspondence. The first is that it only works for extremal and near extremal black holes [63, 64]. The second is that very little is known about the dual CFT. Basically what one learns now is the central charge of the CFT, but one knows nothing about the nature of the quantum states making up the system. So it will be particularly interesting to generalize the calculation to non-extremal black holes, and to understand better about the dual conformal field theory.

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APPENDIX A

SOME BASICS WITH CURVED SPACETIMES

To describe the properties of a curved spacetime, one wants the formulation to be independent of how the spacetime manifold is parameterized. For this purpose, the usual partial derivative need to be covariantized, which is done with the help of a connection. For example, the covariant derivative on a vector field ξ is

$$\nabla_\mu \xi_\nu = \partial_\mu \xi_\nu - \Gamma_{\mu\nu}^\rho \xi_\rho. \quad (\text{A.1})$$

The connection can be uniquely determined, when it is metric compatible ($\nabla_\rho g_{\mu\nu} = 0$) and symmetric,

$$\Gamma_{\mu\nu}^\rho = \Gamma_{\nu\mu}^\rho = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right). \quad (\text{A.2})$$

The most important quantity to describe the geometry of a curved spacetime is the Riemann tensor, which is defined by

$$\begin{aligned} \left[\nabla_\mu, \nabla_\nu \right] \xi_\rho &= R_{\mu\nu\rho}{}^\sigma \xi_\sigma, \\ R^\rho{}_{\sigma\mu\nu} &= \partial_\mu \Gamma_{\nu\sigma}^\rho - \partial_\nu \Gamma_{\mu\sigma}^\rho + \Gamma_{\mu\lambda}^\rho \Gamma_{\sigma\nu}^\lambda - \Gamma_{\nu\lambda}^\rho \Gamma_{\sigma\mu}^\lambda. \end{aligned} \quad (\text{A.3})$$

The Riemann tensor has some nice properties,

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho} = R_{\rho\sigma\mu\nu}, \quad (\text{A.4})$$

$$R_{\mu[\nu\rho\sigma]} = R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0, \quad (\text{A.5})$$

$$R_{\mu\nu[\rho\sigma;\lambda]} = R_{\mu\nu\rho\sigma;\lambda} + R_{\mu\nu\sigma\lambda;\rho} + R_{\mu\nu\lambda\rho;\sigma} = 0. \quad (\text{A.6})$$

From the Riemann tensor, one can define the Ricci tensor, $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} = R_{\nu\mu}$, and

the Riemann scalar, $R = g^{\mu\nu} R_{\mu\nu}$. By multiplying (A.6) with $g^{\mu\rho} g^{\nu\lambda}$, one gets

$$2R^\mu_{\sigma;\mu} - R_{;\sigma} = 0 \implies \nabla_\mu \left(R^{\mu\nu} - \frac{R}{2} g^{\mu\nu} \right) = 0. \quad (\text{A.7})$$

Sometimes it is convenient to use differential forms, such as a p -form,

$$\mathbf{w}_p = \frac{1}{p!} w_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}. \quad (\text{A.8})$$

Its Hodge-* dual is defined by (note $|\epsilon_{\dots}| = \sqrt{|g|}$)

$$*\mathbf{w}_p = w^{\mu_1 \dots \mu_p} \frac{1}{p!(n-p)!} \epsilon_{\mu_1 \dots \mu_p \nu_1 \dots \nu_{n-p}} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_{n-p}}. \quad (\text{A.9})$$

One can also write it as

$$*\mathbf{w}_p = (d^{n-p}x)_{\mu_1 \dots \mu_p} w^{\mu_1 \dots \mu_p}, \quad (\text{A.10})$$

$$(d^{n-p}x)_{\mu_1 \dots \mu_p} = \frac{1}{p!(n-p)!} \epsilon_{\mu_1 \dots \mu_p \nu_1 \dots \nu_{n-p}} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_{n-p}}. \quad (\text{A.11})$$

With this, Stokes's theorem $\int_\Sigma d * \mathbf{w}_p = \oint_{\partial\Sigma} * \mathbf{w}_p$ can be written as

$$\int_\Sigma (d^{n-p+1}x)_{\mu_2 \dots \mu_p} \nabla_{\mu_1} w^{\mu_1 \mu_2 \dots \mu_p} = \oint_{\partial\Sigma} (d^{n-p}x)_{\mu_2 \dots \mu_p \mu_1} w^{\mu_1 \mu_2 \dots \mu_p}. \quad (\text{A.12})$$

Now if there is a conserved current, $\nabla_\mu \mathcal{J}^\mu = 0$, one has

$$\begin{aligned} 0 &= \int_{\mathcal{M}} \sqrt{|g|} d^n x \nabla_\mu \mathcal{J}^\mu = \oint_{\partial\mathcal{M}} (d^{n-1}x)_\mu \mathcal{J}^\mu \\ &= \left(\oint_{\Sigma_{t_2}} - \oint_{\Sigma_{t_1}} \right) (d^{n-1}x)_\mu \mathcal{J}^\mu, \end{aligned} \quad (\text{A.13})$$

where Σ_{t_1} and Σ_{t_2} are the two space-like boundaries of the manifold \mathcal{M} . We have assumed that the contribution from other boundaries vanishes. As a result, a conserved charge can be defined to be

$$Q = \oint_{\Sigma_{t_1}} (d^{n-1}x)_\mu \mathcal{J}^\mu = \oint_{\Sigma_{t_2}} (d^{n-1}x)_\mu \mathcal{J}^\mu, \quad (\text{A.14})$$

where one can replace Σ_{t_1} or Σ_{t_2} with any other space-like hypersurface.

For a function defined in a curved spacetime, its variation from one point to another is given by the Lie derivative, which is always defined with the help of a vector field indicating the displacement. For example, the Lie derivatives for a scalar field, a vector field and a rank two tensor are given by

$$\begin{aligned}\mathcal{L}_\xi \phi &= \xi^\mu \partial_\mu \phi, \quad \mathcal{L}_\xi A_\mu = \xi^\nu \partial_\nu A_\mu + A_\nu \partial_\mu \xi^\nu, \\ \mathcal{L}_\xi T_{\mu\nu} &= \xi^\rho \partial_\rho T_{\mu\nu} + T_{\rho\nu} \partial_\mu \xi^\rho + T_{\mu\rho} \partial_\nu \xi^\rho,\end{aligned}\tag{A.15}$$

where the partial derivatives are equivalent to the covariant ones. For a general differential form, the Lie derivative is given by

$$\mathcal{L}_\xi \mathbf{w}_p = i_\xi d\mathbf{w}_p + d(i_\xi \mathbf{w}_p),\tag{A.16}$$

where the contraction i_ξ is defined as

$$i_\xi \mathbf{w}_p = \xi^\mu \frac{w_{\mu\mu_1\cdots\mu_{p-1}}}{(p-1)!} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p-1}}.\tag{A.17}$$

In a curved spacetime, the metric is different everywhere in general. But if it is invariant along a direction characterized by the vector ξ , then

$$\mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0.\tag{A.18}$$

The invariance of the metric means a symmetry of the spacetime. Now the corresponding vector ξ is called a Killing vector. For a Killing vector,

$$\begin{aligned}\nabla_\mu \nabla_\nu \xi_\alpha &= \frac{1}{2} \left(\nabla_\mu \nabla_\nu \xi_\alpha - \nabla_\mu \nabla_\alpha \xi_\nu \right) \\ &= \frac{1}{2} \left(\left[\nabla_\mu, \nabla_\nu \right] \xi_\alpha - \left[\nabla_\mu, \nabla_\alpha \right] \xi_\nu + \nabla_\nu \nabla_\mu \xi_\alpha - \nabla_\alpha \nabla_\mu \xi_\nu \right) \\ &= \frac{1}{2} \left(\left[\nabla_\mu, \nabla_\nu \right] \xi_\alpha - \left[\nabla_\mu, \nabla_\alpha \right] \xi_\nu - \left[\nabla_\nu, \nabla_\alpha \right] \xi_\mu \right)\end{aligned}$$

$$= \frac{1}{2} \left(R_{\mu\nu\alpha\beta} - R_{\mu\alpha\nu\beta} - R_{\nu\alpha\mu\beta} \right) \xi^\beta = R_{\alpha\nu\mu\beta} \xi^\beta. \quad (\text{A.19})$$

As a byproduct of this,

$$R_{\nu\beta} \xi^\beta = R^\mu_{\nu\mu\beta} \xi^\beta = \nabla_\mu \nabla_\nu \xi^\mu. \quad (\text{A.20})$$

What's more,

$$\begin{aligned} \nabla^\mu \nabla_\rho \nabla_\mu \xi^\rho &= -\nabla^\mu \nabla_\rho \nabla^\rho \xi_\mu = -\nabla_\mu \square \xi^\mu, \\ \text{and } \nabla^\mu \nabla_\rho \nabla_\mu \xi^\rho &= \left(\nabla_\rho \nabla^\mu + [\nabla^\mu, \nabla_\rho] \right) \nabla_\mu \xi^\rho \\ &= \nabla_\rho \square \xi^\rho + R^\mu_{\rho\mu}{}^\sigma \nabla_\sigma \xi^\rho + R^\mu_{\rho}{}^\rho{}_\sigma \nabla_\mu \xi^\sigma \\ &= \nabla_\rho \square \xi^\rho + R_{\rho\sigma} \nabla^\sigma \xi^\rho - R_{\mu\sigma} \nabla^\mu \xi^\sigma \\ &= \nabla_\rho \square \xi^\rho, \\ \implies \nabla^\mu \nabla_\rho \nabla_\mu \xi^\rho &= 0. \end{aligned} \quad (\text{A.21})$$

From (A.20) and (A.21),

$$\nabla^\mu (R_{\mu\nu} \xi^\nu) = \nabla^\mu \nabla_\rho \nabla_\mu \xi^\rho = 0. \quad (\text{A.22})$$

Using (A.7), one gets that

$$\xi^\mu \partial_\mu R = 0. \quad (\text{A.23})$$

This is reasonable since all the geometry properties (including the Riemann scalar!) should be invariant along the direction of a Killing vector.

APPENDIX B

CALCULATING THE CENTRAL TERM $K[\xi, \zeta]$

The central term $K[\xi_m^a, \xi_n^a]$ for (3.147) can be calculated by using (3.100) and (3.111). To do that, let's first write down the non-vanishing metric elements in (3.147),⁸

$$\begin{aligned}
 G_{tt} &= -A(1 + r^2) + k^2 r^2, \\
 G_{at} &= G_{ta} = k_a r, \\
 G_{ab} &= g_{ab}^0, \\
 G_{ij} &= g_{ij}^0, \\
 G_{rr} &= \frac{A}{1 + r^2},
 \end{aligned} \tag{B.1}$$

with $k_a = g_{ab}^0 k^b$, $k^2 = g_{ab}^0 k^a k^b$ and $A = 2f_r^0/\Delta_0''$. Note $f_r^0 = f_r(r_H, \theta^i)$, $g_{ij}^0 = g_{ij}(r_H, \theta^i)$ and $g_{ab}^0 = g_{ab}(r_H, \theta^i)$ are functions of θ^i 's only, while $\Delta_0'' = \Delta''(r_H)$ and k^a 's are constant. Let (g^{0ab}) be the inverse of (g_{ab}^0) , and (g^{0ij}) be the inverse of (g_{ij}^0) , one has

$$\begin{aligned}
 G^{tt} &= -\frac{1}{A(1 + r^2)}, \\
 G^{at} &= G^{ta} = \frac{k^a r}{A(1 + r^2)}, \\
 G^{ab} &= g^{0ab} - \frac{k^a k^b r^2}{A(1 + r^2)}, \\
 G^{ij} &= g^{0ij}, \\
 G^{rr} &= \frac{1 + r^2}{A}.
 \end{aligned} \tag{B.2}$$

⁸Here we shall use the capital letter G to denote the full metric (3.147), in order to distinguish it from the elements g_{ij}^0 and g_{ab}^0 .

For later convenience, note that

$$\begin{aligned}
\Gamma_{ra}^t &= -\frac{1}{2A(1+r^2)}k_a, \\
\Gamma_{rt}^t &= \frac{r}{1+r^2} - \frac{k^2 r}{2A(1+r^2)}, \\
\Gamma_{rr}^r &= -\frac{r}{1+r^2}, \\
\Gamma_{rb}^a &= \frac{r}{2A(1+r^2)}k^a k_b, \\
\Gamma_{rj}^i &= 0, \\
\Gamma_{rr}^t &= 0, \\
\Gamma_{rt}^a &= \frac{1-r^2}{2(1+r^2)}k^a + \frac{k^2 r^2}{2A(1+r^2)}k^a.
\end{aligned} \tag{B.3}$$

Given a particular azimuthal angle $\phi^{\bar{a}}$, and the Killing vector

$$\xi_n = -e^{-in\phi^{\bar{a}}} \partial_{\phi^{\bar{a}}} - inr e^{-in\phi^{\bar{a}}} \partial_r, \tag{B.4}$$

the nontrivial elements of

$$h_{\mu\nu}(\xi_n) = \mathcal{L}_{\xi_n} G_{\mu\nu} = \xi_n^\rho \partial_\rho G_{\mu\nu} + G_{\mu\rho} \partial_\nu \xi_n^\rho + G_{\rho\nu} \partial_\mu \xi_n^\rho \tag{B.5}$$

are given by

$$\begin{aligned}
h_{rr} &= \xi_n^r \partial_r G_{rr} + 2G_{rr} \partial_r \xi_n^r = -\frac{2ine^{-in\phi^{\bar{a}}} A}{(1+r^2)^2}, \\
h_{ra} &= G_{rr} \partial_a \xi_n^r = -\frac{n^2 r e^{-in\phi^{\bar{a}}} A}{1+r^2} \delta_{a\bar{a}}, \\
h_{tt} &= \xi_n^r \partial_r G_{tt} = 2inr^2 e^{-in\phi^{\bar{a}}} (A - k^2), \\
h_{ta} &= \xi_n^r \partial_r G_{ta} + G_{tb} \partial_a \xi_n^b = -inr e^{-in\phi^{\bar{a}}} (k_a - k_{\bar{a}} \delta_{\bar{a}a}), \\
h_{ab} &= G_{ac} \partial_b \xi_n^c + G_{cb} \partial_a \xi_n^c = ine^{-in\phi^{\bar{a}}} (g_{a\bar{a}}^0 \delta_{\bar{a}b} + g_{b\bar{a}}^0 \delta_{\bar{a}a}).
\end{aligned} \tag{B.6}$$

As a result, $h = 0$ and

$$\begin{aligned}
h^{rr} &= G^{rr} G^{rr} h_{rr} = -\frac{2ine^{-in\phi^{\bar{a}}}}{A}, \\
h^{ra} &= G^{rr} G^{ab} h_{rb} = -n^2 r e^{-in\phi^{\bar{a}}} \left(g^{0a\bar{a}} - \frac{r^2 k^a k^{\bar{a}}}{A(1+r^2)} \right), \\
h^{rt} &= G^{rr} G^{ta} h_{ra} = -\frac{n^2 r^2 e^{-in\phi^{\bar{a}}}}{A(1+r^2)} k^{\bar{a}}, \\
h^{tt} &= G^{tt} G^{tt} h_{tt} + 2G^{tt} G^{ta} h_{ta} + G^{ta} G^{tb} h_{ab} = \frac{2inr^2 e^{-in\phi^{\bar{a}}}}{A(1+r^2)^2}, \\
h^{ta} &= G^{tt} G^{at} h_{tt} + (G^{tt} G^{ab} + G^{tb} G^{at}) h_{tb} + G^{tb} G^{ac} h_{bc} \\
&= \frac{inr e^{-in\phi^{\bar{a}}}}{A(1+r^2)} \left(\frac{1-r^2}{1+r^2} k^a + k^{\bar{a}} \delta^{\bar{a}a} \right), \\
h^{ab} &= G^{at} G^{bt} h_{tt} + (G^{at} G^{bc} + G^{ac} G^{bt}) h_{tc} + G^{ac} G^{bd} h_{cd} \\
&= ine^{-in\phi^{\bar{a}}} \left[\delta^{a\bar{a}} g^{0b\bar{a}} + \delta^{b\bar{a}} g^{0a\bar{a}} - \frac{2r^2 k^a k^b}{A(1+r^2)^2} \right. \\
&\quad \left. - \frac{r^2 k^{\bar{a}} (\delta^{a\bar{a}} k^b + \delta^{b\bar{a}} k^a)}{A(1+r^2)} \right]. \tag{B.7}
\end{aligned}$$

From (3.111), one has

$$\begin{aligned}
k^{rt} &= \xi_m^t \nabla^r h - \xi_m^t \nabla_\rho h^{r\rho} + \frac{h}{2} \nabla^t \xi_m^r - h^{t\rho} \nabla_\rho \xi_m^r + \xi_{m\rho} \nabla^t h^{r\rho} \\
&\quad - \xi_m^r \nabla^t h + \xi_m^r \nabla_\rho h^{t\rho} - \frac{h}{2} \nabla^r \xi_m^t + h^{r\rho} \nabla_\rho \xi_m^t - \xi_{m\rho} \nabla^r h^{t\rho}. \tag{B.8}
\end{aligned}$$

We are only interested in terms that will lead to m^3 when $m+n=0$ is applied,

$$\begin{aligned}
\xi_m^r \nabla_\rho h^{t\rho} &= \xi_m^r (\partial_\rho h^{t\rho} + \Gamma_{\rho\sigma}^t h^{\sigma\rho} + \Gamma_{\rho\sigma}^\rho h^{t\sigma}) \\
&\approx \xi_m^r (\partial_{\bar{a}} h^{t\bar{a}} + \partial_r h^{tr} + 2\Gamma_{ra}^t h^{ra} + 2\Gamma_{rt}^t h^{rt} + \Gamma_{\rho r}^\rho h^{tr}), \\
&= \frac{imn^2 r^2 e^{-i(m+n)\phi^{\bar{a}}}}{2A(1+r^2)} \left(\frac{2r^2-2}{1+r^2} \right) k^{\bar{a}}, \\
-h^{t\rho} \nabla_\rho \xi_m^r &= -h^{t\rho} (\partial_\rho \xi_m^r + \Gamma_{\rho\sigma}^r \xi_m^\sigma) \\
&\approx -h^{t\bar{a}} \partial_{\bar{a}} \xi_m^r - h^{tr} (\partial_r \xi_m^r + \Gamma_{rr}^r \xi_m^r) \\
&= \frac{imn^2 r^2 e^{-i(m+n)\phi^{\bar{a}}}}{2A(1+r^2)} \left(\frac{4m/n-2}{1+r^2} \right) k^{\bar{a}},
\end{aligned}$$

$$\begin{aligned}
h^{r\rho}\nabla_\rho\xi_m^t &= h^{r\rho}(\partial_\rho\xi_m^t + \Gamma_{\rho\sigma}^t\xi_m^\sigma) \\
&\approx (h^{ra}\Gamma_{ar}^t + h^{rt}\Gamma_{tr}^t)\xi_m^r \\
&= \frac{imn^2r^2e^{-i(m+n)\phi^{\bar{a}}}}{2A(1+r^2)}\left(\frac{r^2-1}{1+r^2}\right)k^{\bar{a}}, \\
\xi_{m\rho}\nabla^t h^{r\rho} &= \xi_m^\rho G^{rr}(G^{tt}\nabla_t h_{r\rho} + G^{ta}\nabla_a h_{r\rho}) \\
&= \xi_m^\rho G^{rr}G^{tt}(\partial_t h_{r\rho} - \Gamma_{tr}^\sigma h_{\sigma\rho} - \Gamma_{t\rho}^\sigma h_{r\sigma}) \\
&\quad + \xi_m^\rho G^{rr}G^{ta}(\partial_a h_{r\rho} - \Gamma_{ar}^\sigma h_{\sigma\rho} - \Gamma_{a\rho}^\sigma h_{r\sigma}) \\
&\approx \xi_m^r G^{rr}G^{tt}(-\Gamma_{tr}^{\bar{a}}h_{\bar{a}r} - \Gamma_{tr}^{\bar{a}}h_{r\bar{a}}) \\
&\quad + \xi_m^{\bar{a}} G^{rr}G^{t\bar{a}}\partial_{\bar{a}}h_{r\bar{a}} + \xi_m^r G^{rr}G^{t\bar{a}}\partial_{\bar{a}}h_{rr} \\
&\quad + \xi_m^r G^{rr}G^{ta}(-\Gamma_{ar}^{\bar{a}}h_{\bar{a}r} - \Gamma_{ar}^{\bar{a}}h_{r\bar{a}}) \\
&= \frac{imn^2r^2e^{-i(m+n)\phi^{\bar{a}}}}{2A(1+r^2)}\left(\frac{6-2r^2}{1+r^2} - \frac{2n}{m}\right)k^{\bar{a}}, \\
-\xi_{m\rho}\nabla^r h^{t\rho} &= -\xi_{m\rho}G^{rr}(\partial_r h^{t\rho} + \Gamma_{r\sigma}^t h^{\sigma\rho} + \Gamma_{r\sigma}^\rho h^{t\sigma}) \\
&\approx -\xi_{mr}G^{rr}(\partial_r h^{tr} + \Gamma_{rt}^t h^{tr} + \Gamma_{r\bar{a}}^t h^{\bar{a}r} + \Gamma_{rr}^r h^{tr}) \\
&= \frac{imn^2r^2e^{-i(m+n)\phi^{\bar{a}}}}{2A(1+r^2)}\left(1 - \frac{4}{1+r^2}\right)k^{\bar{a}}, \tag{B.9}
\end{aligned}$$

where “ \approx ” means only terms contributing to m^3 are preserved. The integral in (3.100) is done at $r \rightarrow +\infty$. In this limit, we have from (B.8) and (B.9),

$$k^{rt} = \frac{i(m-n)n^2e^{-i(m+n)\phi^{\bar{a}}}}{A}k^{\bar{a}}. \tag{B.10}$$

Now using (3.100) and (3.111), and noticing that

$$\begin{aligned}
\oint (d^{d-2}x)_{\mu\nu} k^{\mu\nu} &= \oint 2(d^{d-2}x)_{rt} k^{rt}, \\
(d^{d-2}x)_{rt} &= \frac{1}{2}A\sqrt{|g_{ij}^0|}\sqrt{|g_{ab}^0|}\prod_i d\theta^i \prod_a d\phi^a, \tag{B.11}
\end{aligned}$$

one has

$$\begin{aligned}
K[\xi_m^{\bar{a}}, \xi_n^{\bar{a}}] &= -\frac{i(m-n)n^2 k^{\bar{a}}}{16\pi} \oint \sqrt{|g_{ij}^0|} \sqrt{|g_{ab}^0|} \prod_i d\theta^i \prod_a d\phi^a e^{-i(m+n)\phi^{\bar{a}}} \\
&= -\frac{i(m-n)n^2 k^{\bar{a}}}{16\pi} \delta_{m+n} \mathcal{A}_{rea}.
\end{aligned} \tag{B.12}$$

Note $\mathcal{A}_{rea} = \oint \sqrt{|g_{ij}^0|} \sqrt{|g_{ab}^0|} \prod_i d\theta^i \prod_a d\phi^a$ is the horizon area for (3.127).

VITA

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